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The Motive of the Classifying Stack of the Orthogonal Group

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Abstract. We compute the motive of the classifying stack of an orthogonal group in the Grothendieck ring of stacks over a field of characteristic different from two.

1. Introduction

The Grothendieck ring of stacks over a field k has been introduced by a number of authors [1; 6; 8; 13]. Denote this ring by $\hat{K}_0(\mathrm{Var}_k)$. An algebraic group G defined over k is called special if any G-torsor over a k-variety is locally trivial in the Zariski topology. General linear, special linear, and symplectic groups are special. Special orthogonal groups are not special in dimensions greater than two. Serre [11] proved that special groups are linear and connected. Over algebraically closed fields, the special groups were classified by Grothendieck [7].

For a special group G, the motive $[G]$ is invertible in $\hat{K}_0(\mathrm{Var}_k)$, and its inverse is equal to the motive of the classifying stack BG. This naturally raises the problem of computing the motive of BG when the group G is not special. For finite group schemes, a number of examples were computed in [5]. The case of groups of positive dimension is more difficult. In [3] it was shown that $[BPGL_n] = [PGL_n]^{-1}$ for $n = 2$ or 3 with mild restrictions on the field k.

The main result of this paper, Theorem 3.7, computes the motive of the classifying stack of an orthogonal group over a field whose characteristic is not two. In odd dimensions the result is that the motive is equal to the inverse of the motive of the split special orthogonal group in the same dimension. To prove Theorem 3.7, we first compute the motive of the variety of nondegenerate quadratic forms of fixed dimension. This motive was already computed in [2], using results of [9]. Our computation is different, relying on generating function techniques. Using Theorem 3.7, we are able to compute the motives of classifying stacks of the special orthogonal groups in odd dimensions.

1.1. Notation

We will work over a base field k with $\mathrm{char}(k) \neq 2$. If n is a nonnegative integer, then we denote by $[n]_{\mathbb{L}}$ the nth Gaussian polynomial in the Lefschetz motive $\mathbb{L}$. Explicitly,

$$[n]_{\mathbb{L}} = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1}.$$

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