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Fractional Integrals and Riesz Transforms Acting on Certain Lipschitz Spaces

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Abstract. We make a unifying approach to the study of mapping properties of fractional integrals and Riesz transforms acting on spaces of functions f satisfying

$$\sup_B\left(\frac{1}{w(a,r)}\left(\frac{1}{|B|}\int_B|f-m_Bf|^q\right)^{1/q}\right)<\infty,$$

where w is a nonnegative functional defined on the family of balls $B\subset\mathbb{R}^n$ with center a and radius r. So, at the same time, we are able to treat such cases as BMO, Lipschitz spaces, and spaces of functions with variable smoothness among others. Results about pointwise smoothness related to these spaces are included as well.

1. Introduction

Let $w:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ be a measurable function. For given $1\le q<\infty$, we define the space $\mathrm{BMO}_{w,q}$ as the set of locally integrable functions f on $\mathbb{R}^n$ such that

$$\frac{1}{w(a,r)}\left(\frac{1}{|B|}\int_B|f(x)-m_Bf|^q\,dx\right)^{1/q}\le C \tag{1.1}$$

for some $C>0$ and for every ball $B\subset\mathbb{R}^n$ with center a and radius r, where m_Bf is the average of f over B, namely $m_Bf=|B|^{-1}\int_B f(y)\,dy$. As it can be easily seen, the expression

$$\|f\|_{w,q}=\sup_{B\subset\mathbb{R}^n}\left\{\frac{1}{w(a,r)}\left(\frac{1}{|B|}\int_B|f(x)-m_Bf|^q\,dx\right)^{1/q}\right\}$$

turns out to be a seminorm for this space. Then, $\mathrm{BMO}_{w,q}$ modulo constants is a Banach space. The space $\mathrm{BMO}_{w,1}$ was introduced by Nakai and Yabuta [19], although a version defined on the n-dimensional torus had already appeared in Janson [14] in connection with the identification of pointwise multipliers of the space of functions with mean oscillation controlled by a positive, nondecreasing function φ, that is, BMO_φ (see [23]). The general $\mathrm{BMO}_{w,q}$, $1\le q<\infty$, was introduced in [20], where a complete study on their pointwise multipliers is done.

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