Michigan Math. J. 64 (2015), 721–758

On Some Varieties Associated with Trees

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Abstract. We consider some affine algebraic varieties attached to finite trees and closely related to cluster algebras. Their definition involves a canonical coloring of vertices of trees into three colors. These varieties are proved to be smooth and to admit sometimes free actions of algebraic tori. Some results are obtained on their number of points over finite fields and on their cohomology.

Introduction

The theory of cluster algebras, introduced by S. Fomin and A. Zelevinsky around 2000 [FZ02; FZ03], was motivated initially by the study of total positivity in Lie groups and canonical bases in quantum groups. It has since then developed rapidly in many directions, among which we can cite (for example) triangulated categories [BMR+06], triangulations of surfaces [FST08], and Poisson geometry [GSV03; GSV10].

Because cluster algebras are commutative algebras endowed with more structure, it is natural to study them from the point of view of algebraic geometry. The geometric study of cluster algebras has nevertheless been mostly concentrated on aspects related to Poisson geometry or symplectic geometry. The appearance of the known cluster structure on coordinate rings of Grassmannians in a physical context [ABC+12] has raised recently the interest in the computation of integrals on the varieties associated with cluster algebras. The natural context for this is of course the cohomology ring.

In the present article, we aim to study some varieties closely related to the spectrum of cluster algebras and their cohomology rings. General cluster algebras are defined using a quiver or a skew-symmetric matrix. For our purposes, as a starting point, we need a presentation by generators and relations of the cluster algebras. This is available for cluster algebras with an acyclic quiver [BFZ05] and in a few other cases (see, e.g., [Mul13]). The choice has been made here to restrict to a still smaller class, namely cluster algebras with a quiver that is a tree, in the hope that the answers may be simpler in that case, and also because all finite Dynkin diagrams are trees.

Cluster algebras come with a subalgebra generated by so-called frozen (or coefficient) variables, which are invertible elements. This corresponds to a morphism from the spectrum of the cluster algebra to an algebraic torus. At the start

Received April 1, 2014. Revision received June 17, 2015.
This work has been supported by the ANR program CARMA (ANR-12-BS01-0017-02).