Michigan Math. J. 64 (2015), 675–719

Apolarity and Direct Sum Decomposability of Polynomials

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Abstract. A polynomial is a direct sum if it can be written as a sum of two nonzero polynomials in some distinct sets of variables, up to a linear change of variables. We analyze criteria for a homogeneous polynomial to be decomposable as a direct sum in terms of the apolar ideal of the polynomial. We prove that the apolar ideal of a polynomial of degree d strictly depending on all variables has a minimal generator of degree d if and only if it is a limit of direct sums.

1. Introduction

A homogeneous polynomial F is a *direct sum* if there exist nonzero polynomials F_1, F_2 such that $F = F_1 + F_2$ and $F_1 = F_1(t_1, \dots, t_s)$, $F_2 = F_2(t_{s+1}, \dots, t_n)$ for some linearly independent linear forms $t_1, \dots, t_n$. For example, $F = xy$ is a direct sum since $F = \frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$. In coordinate-free terms, $F \in S^d V$ is a direct sum if $F = F_1 + F_2$ for nonzero $F_i \in S^d V_i$, $i = 1, 2$, such that $V_1 \oplus V_2 = V$.

Most polynomials are not direct sums; see Lemma 3.3. Nevertheless, it can be difficult to show that a particular polynomial is not a direct sum. For instance, S. Shafiei shared with us the following question: is the generic determinant $\det_n = \det((x_{i,j})_{i,j=1}^n)$, a homogeneous form of degree n in n^2 variables, a direct sum? For $n = 2$, $\det_2 = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$ is visibly a direct sum. On the other hand, for $n > 2$, it is easy to see that the determinant is not decomposable as a direct sum in the original variables, but it is not immediately clear whether it is decomposable after a linear change of coordinates. We answer this question in the negative; see Corollary 1.2.

Problem A. Give necessary or sufficient conditions for a polynomial to be a direct sum.

Received August 10, 2013. Revision received July 14, 2015.

W. Buczyńska is supported by the research project “Rangi i rangi brzegowe wielomianów oraz równania rozmaitości siecznych” funded by Polish Financial Means for Science in 2012–2014. J. Buczyński is supported by the project “Secant varieties, computational complexity, and toric degenerations” realized within the Homing Plus programme of Foundation for Polish Science, cofinanced from European Union, Regional Development Fund. J. Buczyński is also supported by the scholarship “START” of the Foundation for Polish Science. Furthermore, while revising the article, he is supported by a scholarship of the Polish Ministry of Science.