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Characteristic Classes for Curves of Genus One

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Abstract. We compute the cohomology of the stack $\mathcal{M}_1$ over $\mathbf{C}$ with coefficients in $\mathbf{Z}[\frac{1}{2}]$, and in low degrees with coefficients in $\mathbf{Z}$. Cohomology classes on $\mathcal{M}_1$ give rise to *characteristic classes,* cohomological invariants of families of curves of genus one. We prove a number of vanishing results for those characteristic classes and give explicit examples of families with nonvanishing characteristic classes.

1. Introduction and Statement of the Results

1.1. The Cohomology of $\mathcal{M}_1$

We denote by $\mathcal{M}_1$ the algebraic stack of curves of genus one and by $\mathcal{M}_{1,1}$ the algebraic stack of elliptic curves, both over $\mathbf{C}$. See Section 2 for more details. If X is an algebraic stack of finite type over $\mathbf{C}$, then we denote by X^{an} its analytification and by $\mathrm{H}^\bullet(X^{\mathrm{an}}, -)$ its singular cohomology.

Consider the map $J\colon \mathcal{M}_1 \to \mathcal{M}_{1,1}$, sending a curve to its Jacobian, and its Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathcal{M}_{1,1}^{\mathrm{an}}, \mathrm{R}^q J_* \mathbf{Z}) \Longrightarrow \mathrm{H}^{p+q}(\mathcal{M}_1^{\mathrm{an}}, \mathbf{Z}). \tag{1}$$

The fibers of J are classifying spaces of rank two tori, and we have

$$\mathrm{R}^q J_* \mathbf{Z} = \begin{cases} \mathrm{Sym}^k \mathrm{R}^1 \pi_* \mathbf{Z} & (q = 2k), \\ 0 & (q \text{ odd}), \end{cases}$$

where $\pi\colon \mathcal{E} \to \mathcal{M}_{1,1}$ is the universal elliptic curve. The upper half-plane is contractible, and since it is a universal covering of $\mathcal{M}_{1,1}^{\mathrm{an}}$ with covering group $\mathrm{SL}_2 \mathbf{Z}$, the cohomology of local systems on $\mathcal{M}_{1,1}$ can be expressed in terms of group cohomology for $\mathrm{SL}_2 \mathbf{Z}$. We find:

$$E_2^{p,q} = \begin{cases} \mathrm{H}^p(\mathrm{SL}_2 \mathbf{Z}, \mathrm{Sym}^k(\mathbf{Z}^2)) & (q = 2k), \\ 0 & (q \text{ odd}), \end{cases}$$

where $\mathbf{Z}^2$ is the standard representation of $\mathrm{SL}_2 \mathbf{Z}$.

The group $\mathrm{SL}_2 \mathbf{Z}$ has a free subgroup of index 12, so if M is an $\mathrm{SL}_2 \mathbf{Z}$-module on which 6 is invertible, then $\mathrm{H}^\bullet(\mathrm{SL}_2 \mathbf{Z}, M)$ is concentrated in degrees 0 and 1. It immediately follows that the spectral sequence (1) tensored with $\mathbf{Z}[\frac{1}{6}]$ degenerates at E_2. We show that it degenerates already with $\mathbf{Z}[\frac{1}{2}]$-coefficients and, moreover, that the filtration on cohomology splits. In other words:

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