Michigan Math. J. 64 (2015), 569–586

Strange Duality for Height Zero Moduli Spaces of Sheaves on $\mathbb{P}^2$

Takeshi Abe

1. Introduction

Strange duality is a duality between vector spaces of global sections of line bundles on moduli spaces of sheaves. Originally it was studied for moduli spaces of vector bundles (or principal bundles) on curves (see [P]). In this paper we consider the strange duality for moduli spaces of sheaves on $\mathbb{P}^2$.

Result

For a coherent sheaf E of positive rank r on $\mathbb{P}^2$, we define the rational numbers $\mu(E)$ and $\Delta(E)$, called the slope and the discriminant, respectively, by

$$\mu(E) = \frac{c_1(E)}{r},$$
$$\Delta(E) = \frac{1}{r}\left(c_2(E) - \frac{r-1}{2r}c_1(E)^2\right).$$

For a positive integer r and rational numbers s, d, we denote by $M(r,s,d)$ the moduli space of rank r semistable sheaves E on $\mathbb{P}^2$ with $\mu(E) = s$ and $\Delta(E) = d$.

We recall the definition of a strange duality map. Fix positive integers r, r' and rational numbers s, s', d, d' such that $\chi(E \otimes E') = 0$ for $E \in M := M(r,s,d)$ and $E' \in M' := M(r',s',d')$. Assume that $s + s' \geq 0$ (so that we have $H^2(E \otimes E') = 0$).

Consider the locus

$$\Theta := \{(E, E') \mid H^0(E \otimes E') \neq 0\} \subset M \times M'.$$

If $H^0(E \otimes E') \neq 0$ for all $E \in M$ and $E' \in M'$, then $\Theta = M \times M'$. Now we assume that for some $E \in M$ and $E' \in M'$, we have $H^i(E \otimes E') = 0$, $0 \leq i \leq 2$. In this case, Θ is a divisor on $M \times M'$. The associated line bundle $\mathcal{O}(\Theta)$ is expressed as $\mathcal{D} \boxtimes \mathcal{D}'$ for line bundles $\mathcal{D}$ on M and $\mathcal{D}'$ on M'. By the Kunneth theorem, the section defining the divisor Θ gives rise to a duality map

$$H^0(M', \mathcal{D}')^* \to H^0(M, \mathcal{D}).$$

We call this the strange duality map. The purpose of this paper is to prove the following theorem.

Received October 30, 2014. Revision received January 26, 2015.