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Integer Complexity and Well-Ordering

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Abstract. Define $\|n\|$ to be the *complexity* of n, the smallest number of ones needed to write n using an arbitrary combination of addition and multiplication. John Selfridge showed that $\|n\| \geq 3\log_3 n$ for all n. Define the *defect* of n, denoted $\delta(n)$, to be $\|n\| - 3\log_3 n$. In this paper, we consider the set $\mathscr{D} := \{\delta(n) : n \geq 1\}$ of all defects. We show that as a subset of the real numbers, the set $\mathscr{D}$ is well-ordered, of order type ω^ω. More specifically, for $k \geq 1$ an integer, $\mathscr{D} \cap [0, k)$ has order type ω^k. We also consider some other sets related to $\mathscr{D}$ and show that these too are well-ordered and have order type ω^ω.

1. Introduction

The *complexity* of a natural number n is the least number of 1s needed to write it using any combination of addition and multiplication, with the order of the operations specified using parentheses grouped in any legal nesting. For instance, $n = 11$ has a complexity of 8 since it can be written using eight ones as

$$(1+1+1)(1+1+1)+1+1,$$

but not with any fewer. This notion was implicitly introduced in 1953 by Mahler and Popken [18]; they actually considered the inverse function of the size of the largest number representable using k copies of the number 1. (More generally, they considered the same question for representations using k copies of a positive real number x.) Integer complexity was explicitly studied by John Selfridge and was later popularized by Guy [13; 14]. Following Arias de Reyna [3], we will denote the complexity of n by $\|n\|$.

Integer complexity is approximately logarithmic; it satisfies the bounds

$$3\log_3 n = \frac{3}{\log 3}\log n \leq \|n\| \leq \frac{3}{\log 2}\log n, \quad n > 1. \tag{1.1}$$

The lower bound can be deduced from the result of Mahler and Popken and was explicitly proved by Selfridge [13]. It is attained with equality for $n = 3^k$ for all $k \geq 1$. The upper bound can be obtained by writing n in binary and finding a representation using Horner's algorithm. It is not sharp, and the constant $\frac{3}{\log 2}$ can be improved for large n [22].

The notion of integer complexity is similar in spirit but different in detail from the better known measure of *addition chain length,* which has application to computation of powers, and which is discussed in detail in Knuth [17, Sect. 4.6.3]. One important difference between the two notions is that integer complexity can

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