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On Chow Quotients of Torus Actions

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Abstract. We consider torus actions on Mori dream spaces and ask whether the associated Chow quotient is again a Mori dream space and, if so, what does its Cox ring look like. We provide general tools for the study of these problems and give solutions for $\mathbb{K}^*$-actions on smooth quadrics.

1. Introduction

Consider an action $G \times X \to X$ of a connected linear algebraic group G on a projective variety X defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The Chow quotient is an answer to the problem of associating in a canonical way a quotient to this action: it is defined as the closure of the set of general G-orbit closures viewed as points in the Chow variety; see Section 2 for more background. The Chow quotient always exists, but, in general, its geometry appears to be not easily accessible.

In the present paper, we consider algebraic torus actions $T \times X \to X$ and ask for the Mori dream property of the normalized Chow quotient Y, provided that X is a Mori dream space, that is, has finitely generated Cox ring [14]. A well understood example class is given by subtorus actions on toric varieties. There, the normalized Chow quotient is again toric and hence a Mori dream space. Moreover, the corresponding fan can be computed, and thus the Cox ring of the normalized Chow quotient is accessible as well [16; 6]. Note, however, that there is no hope for comparable statements in general. For example, Castravet and Tevelev [5] showed that the Chow quotient $\overline{M}_{0,n}$ of the maximal torus action on the Grassmannian $G(2, n)$ is not a Mori dream space for n sufficiently large.

Our aim is to provide tools for the treatment of nontoric examples and to open up the case of $\mathbb{K}^*$-actions on smooth projective quadrics as a new example class for positive results. The first main result is the following.

Theorem 1.1. *Let $\mathbb{K}^*$ act on a smooth projective quadric X. Then the associated normalized Chow quotient is a Mori dream space.*

The second result concerns the computation of the Cox ring; recall that the explicit knowledge of the Cox ring is an approach to the geometry of the underlying space [1]. We first prepare and state the result and then discuss the setting. After an equivariant embedding into a projective space and applying a suitable linear

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