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Zeta Functions of Curves with No Rational Points

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Abstract. We show that the motivic zeta functions of smooth, geometrically connected curves with no rational points are rational functions. This was previously known only for curves whose smooth projective models have a rational point on each connected component. In the course of the proof we study the class of a Severi–Brauer scheme over a general base in the Grothendieck ring of varieties.

1. Introduction

Let k be a field, and $K_0(\mathrm{Var}_k)$ the Grothendieck ring of varieties over k. This is the free Abelian group on isomorphism classes $[X]$ of finite-type k-schemes, subject to the following relation:

$$[X] = [Y] + [X \setminus Y] \quad \text{for } Y \hookrightarrow X \text{ a closed embedding.}$$

Multiplication is given by

$$[X] \cdot [Y] = [X \times Y]$$

on classes of finite-type k-schemes and extended bilinearly. This ring was introduced by Grothendieck [12, Letter of 16 August 1964] in a letter to Serre. The Grothendieck ring of varieties is the universal ring through which all "motivic" invariants factor (e.g., Euler characteristic with compact support, Hodge–Deligne polynomial, virtual Chow motive, etc.).

Let $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_k)$ be the class of the affine line.

Example 1. Using the fact that $\mathbb{P}^n = \mathrm{pt} \cup \mathbb{A}^1 \cup \mathbb{A}^2 \cup \cdots \cup \mathbb{A}^n$, we have

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n.$$

In [6, 1.3], Kapranov introduces for each quasi-projective X/k a motivic zeta function $Z_X(t)$.

Definition 2 (Kapranov motivic zeta function). Let X be a quasi-projective k-scheme. Then the motivic zeta function $Z_X(t) \in K_0(\mathrm{Var}_k)[[t]]$ is

$$Z_X(t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n(X)] t^n,$$

where $\mathrm{Sym}^n(X)$ is the quotient of X^n by the symmetric group Σ_n, and Σ_n acts on X^n by permuting the factors. The quotient exists since X is quasi-projective.

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