Michigan Math. J. 64 (2015), 349–381

Koszul Determinantal Rings and $2 \times e$ Matrices of Linear Forms

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Abstract. Let k be an algebraically closed field of characteristic 0. Let X be a $2 \times e$ matrix of linear forms over a polynomial ring $k[\mathsf{x}_1, \dots, \mathsf{x}_n]$ (where $e, n \geq 1$). We prove that the determinantal ring $R = k[\mathsf{x}_1, \dots, \mathsf{x}_n]/I_2(X)$ is Koszul if and only if in any Kronecker–Weierstrass normal form of X, the largest length of a nilpotent block is at most twice the smallest length of a scroll block. As an application, we classify rational normal scrolls whose all section rings by natural coordinates are Koszul. This result settles a conjecture of Conca.

1. Introduction

Let k be an algebraically closed field of characteristic 0, R a commutative, standard graded k-algebra. The last condition means that R is $\mathbb{Z}$-graded, $R_0 = k$, and R is generated as a k-algebra by finitely many elements of degree 1. We say that R is a *Koszul algebra* if k has linear resolution as an R-module. Denote by $\operatorname{reg}_R M$ the Castelnuovo–Mumford regularity of a finitely generated graded R-module M. An equivalent way to express the Koszulness of R is the condition $\operatorname{reg}_R k = 0$. Effective techniques to prove Koszulness include Gröbner deformation, Koszul filtrations, computation of the Betti numbers of k for toric rings, among others. For some survey articles on Koszul algebras, we refer to [11; 16].

In this paper, we study the Koszul property of linear sections of rational normal scrolls. By abuse of terminology, we use "rational normal scrolls" to refer to the homogeneous coordinate rings of the corresponding varieties. These graded algebras are defined by the ideals of 2-minors of some $2 \times e$ matrices of linear forms, where $e \geq 1$. The homogeneous coordinate rings of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^e \to \mathbb{P}^{2e+1}$ and the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^e$ are among the examples; in fact, they are special instances of rational normal scrolls. The rational normal scrolls are a classical and widely studied class of varieties with minimal multiplicity, whose classification is known from works of Del Pezzo and Bertini; see [14]. See also, for example, [2; 3] for some recent works on this topic.

Let X be a $2 \times e$ matrix of linear forms over a polynomial ring $S = k[\mathsf{x}_1, \dots, \mathsf{x}_n]$. Let $R = k[\mathsf{x}_1, \dots, \mathsf{x}_n]/I_2(X)$ be the determinantal ring of X. Algebraic properties of such determinantal rings R were studied in the literature;

Received May 30, 2014. Revision received June 13, 2014.

The first named author is grateful to the support of the Vigoni project (in 2011) and the CARIGE foundation.