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The Symplectic Mapping Class Group of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with $n \le 4$

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Abstract. In this paper, we prove that the Torelli part of the symplectomorphism groups of the n-point ($n \le 4$) blow-ups of the projective plane is trivial. Consequently, we determine the symplectic mapping class group. It is generated by reflections on K_ω-spherical class with zero ω area.

1. Introduction

A symplectic manifold (X, ω) is an even-dimensional manifold X with a closed, nondegenerate two-form ω. The symplectomorphism group of (X, ω), denoted by $\mathrm{Symp}(X, \omega)$, is the group of diffeomorphisms ϕ of M that preserve ω and is given the C^∞-topology. $\mathrm{Symp}(X, \omega)$ is an infinite-dimensional Fréchet Lie group.

For a closed four-dimensional symplectic manifold (X, ω), since Gromov's work [Gro85], the homotopy type of $\mathrm{Symp}(X, \omega)$ has attracted much interest over the past 30 years. For the special case of some monotone 4-manifolds, the (rational) homotopy of $\mathrm{Symp}(X, \omega)$ was fully computed in [Gro85; AM99; Eva11]. However, for an arbitrary symplectic 4-manifold, the complication grows drastically: see [Abr98; AM99; Anj02] for $S^2 \times S^2$ and [AP12] for other instances.

The goal of this note is modest: for some rational 4-manifolds, we compute $\pi_0(\mathrm{Symp}(X, \omega))$, which is the symplectic mapping class group (denoted as SMC for short). In the cases we consider, the homological action of $\mathrm{Symp}(X, \omega)$ is already known in [LW11]. Therefore, it suffices to describe $\pi_0(\mathrm{Symp}_h(X, \omega))$, which is the subgroup of $\mathrm{Symp}(X, \omega)$ acting trivially on homology, namely, its Torelli part.

Theorem 1.1. *$\mathrm{Symp}_h(X, \omega)$ is connected for $X = \mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$ with arbitrary symplectic form ω.*

The cases $S^2 \times S^2$ and $(\mathbb{C}P^2\#k\overline{\mathbb{C}P^2})$ with $k \le 3$ are known before. Our approach actually works in a uniform way for all $k \le 4$ (see discussions in Remark 3.5). We also note that Theorem 1.1 is not true in general for $k \ge 5$; see Seidel's famous example in [Sei08].

Our strategy is based on Evans' beautiful approach in [Eva11] by systematically exploring the geometry of certain stable configuration of symplectic spheres (a related approach first appeared in Abreu's paper [Abr98]). It is summarized by

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