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# On Stable Conjugacy of Finite Subgroups of the Plane Cremona Group, II

Yuri Prokhorov

Abstract. We prove that, except for a few cases, stable linearizability of finite subgroups of the plane Cremona group implies linearizability.

## 1. Introduction

This is a follow-up paper to [BP13]. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Recall that the *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ is the group of birational automorphisms $\mathrm{Bir}(\mathbb{P}^n)$ of the projective space $\mathbb{P}^n$ over $\Bbbk$. Subgroups $G \subset \mathrm{Cr}_n(\Bbbk)$ and $G' \subset \mathrm{Cr}_m(\Bbbk)$ are said to be *stably conjugate* if, for some $N \geq n, m$, they are conjugate in $\mathrm{Cr}_N(\Bbbk)$, where the embeddings $\mathrm{Cr}_n(\Bbbk), \mathrm{Cr}_m(\Bbbk) \subset \mathrm{Cr}_N(\Bbbk)$ are induced by birational isomorphisms $\mathbb{P}^N \dashrightarrow \mathbb{P}^n \times \mathbb{P}^{N-n} \dashrightarrow \mathbb{P}^m \times \mathbb{P}^{N-m}$.

Any embedding of a finite subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is induced by a biregular action on a rational variety $X$. A subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is said to be *linearizable* if one can take $X = \mathbb{P}^n$. A subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is said to be *stably linearizable* if it is stably conjugate to a linear action of $G$ on a vector space $\Bbbk^m$.

The following question is a natural extension of the famous Zariski cancellation problem [BCSD85] to the geometric situation.

Question 1.1. Let $G \subset \mathrm{Cr}_2(\Bbbk)$ be a stably linearizable finite subgroup. Is it true that $G$ is linearizable?

In this paper, we give a partial answer by finding a (very restrictive) list of all subgroups $G \subset \mathrm{Cr}_2(\Bbbk)$ that potentially can give counterexamples to the question.

It is easy to show (see [BP13]) that the group $H^1(G, \mathrm{Pic}(X))$ is a stable birational invariant. In particular, if $G \subset \mathrm{Cr}_n(\Bbbk)$ is stably linearizable, then $H^1(G_1, \mathrm{Pic}(X)) = 0$ for any subgroup $G_1 \subset G$ (then we say that $G \subset \mathrm{Cr}_n(\Bbbk)$ is *$H^1$-trivial*). Any finite subgroup $G \subset \mathrm{Cr}_2(\Bbbk)$ is induced by an action on either a del Pezzo surface or a conic bundle [Isk80]. In the first case, our main result is the following theorem, which is based on a computation of $H^1(G, \mathrm{Pic}(X))$ in [BP13] (see Theorem 2.9).

Theorem 1.2. *Let $X$ be a del Pezzo surface, and let $G \subset \mathrm{Aut}(X)$ be a finite subgroup such that the pair $(X, G)$ is minimal. Then the following are equivalent*:

(i) $H^1(G_1, \mathrm{Pic}(X)) = 0$ *for any subgroup* $G_1 \subset G$,
(ii) *any element of $G$ does not fix a curve of positive genus*,

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