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An Algorithm to Detect Full Irreducibility by Bounding the Volume of Periodic Free Factors

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Abstract. We provide an effective algorithm for determining whether an element ϕ of the outer automorphism group of a free group is fully irreducible. Our method produces a finite list that can be checked for periodic proper free factors.

1. Introduction

Let F be a finitely generated nonabelian free group of rank at least 2. An outer automorphism ϕ is *reducible* if there exists a free factorization $F = A_1 * \cdots * A_k * B$ such that ϕ permutes the conjugacy classes of the A_i; else it is *irreducible.* Although irreducible elements have nice properties, for example, they are known to possess irreducible train-track representatives, irreducibility is not preserved under iteration. Thus, one often considers elements that are *irreducible with irreducible powers* (*iwip*), or *fully irreducible.* These are precisely the outer automorphisms ϕ for which there does not exist a proper free factor $A < F$ whose conjugacy class $[A]$ satisfies $\phi^p([A]) = [A]$ for any $p > 0$. If $\phi^p([A]) = [A]$ for some proper free factor $A < F$ and for some $p > 0$, then we say that $[A]$ is ϕ-periodic and, to avoid cumbersome language, also that the free factor A is ϕ-periodic. Fully irreducible elements are considered analogous to pseudo-Anosov mapping classes of hyperbolic surfaces. As such, they play an important role in the geometry and dynamics of the outer automorphism group $\mathrm{Out}(F)$ of F.

Although considered in some sense a "generic" property in $\mathrm{Out}(F)$, full irreducibility is not generally easy to detect. Kapovich [16] gave an algorithm for determining whether a given $\phi \in \mathrm{Out}(F)$ is fully irreducible, inspired by Pfaff's criterion for full irreducibility in [21]. At points in his algorithm, two processes run simultaneously, and although it is known that one of these must terminate, it is not a priori known which will; it thus seems unclear that the complexity of Kapovich's algorithm can be found without running the algorithm itself.

For mapping class groups and braid groups, there exist algorithms for determining whether or not a given element is pseudo-Anosov [6; 8; 3; 4; 20; 7]. Recently, Koberda and the second author [17] provided an elementary algorithm for determining whether or not a given mapping class is pseudo-Anosov, using a method of "list and check". They show that if a mapping class f is reducible,

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