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Quasi-conformal Maps on Model Filiform Groups

Xiangdong Xie

Abstract. We describe all quasi-conformal maps on the higher (real and complex) model Filiform groups equipped with the Carnot metric, including nonsmooth ones. These maps have very special forms. In particular, they are all bi-Lipschitz and preserve multiple foliations. The results in this paper have implications to the large-scale geometry of nilpotent Lie groups and negatively curved solvable Lie groups.

1. Introduction

In this paper we study quasi-conformal maps on the higher real and complex model Filiform groups equipped with the Carnot metric. We identify all such maps. They are all bi-Lipschitz and preserve multiple foliations. We do not impose any regularity condition on the quasi-conformal maps. However, the group structure forces rigidity and regularity. In particular, in the case of higher complex model Filiform groups, up to taking complex conjugation, all quasi-conformal maps are biholomorphic and in fact affine.

Let K be a field. We only consider the case where K is $\mathbb{R}$ or $\mathbb{C}$. The n-step ($n \geq 2$) model Filiform algebra $\mathfrak{f}_K^n$ over K is an $(n+1)$-dimensional Lie algebra over K. It has a basis $\{e_1, e_2, \dots, e_{n+1}\}$, and the only nontrivial bracket relations are $[e_1, e_j] = e_{j+1}$ for $2 \leq j \leq n$. The Lie algebra $\mathfrak{f}_K^n$ admits a direct sum decomposition of vector subspaces $\mathfrak{f}_K^n = V_1 \oplus \cdots \oplus V_N$, where V_1 is the linear subspace spanned by e_1, e_2, and V_j ($2 \leq j \leq n$) is the linear subspace spanned by e_{j+1}. It is easy to check that $[V_1, V_j] = V_{j+1}$ for $1 \leq j \leq n$, where $V_{n+1} = \{0\}$. Hence, $\mathfrak{f}_K^n$ is a stratified Lie algebra. For $K = \mathbb{R}$ or $\mathbb{C}$, the connected and simply connected Lie group with Lie algebra $\mathfrak{f}_K^n$ will be denoted by F_K^n and is called the n-step model Filiform group over K.

The two-dimensional subspace V_1 of $\mathfrak{f}_{\mathbb{R}}^n$ determines a left-invariant distribution (so called horizontal distribution) on $F_{\mathbb{R}}^n$. On V_1, we consider the inner product with e_1, e_2 as an orthonormal basis. This norm on V_1 then induces a Carnot metric d_c on $F_{\mathbb{R}}^n$. Similarly, the first layer V_1 of $\mathfrak{f}_{\mathbb{C}}^n$ is a four-dimensional real vector subspace spanned by e_1, ie_1, e_2, ie_2 ($i = \sqrt{-1}$), and it determines a left-invariant distribution on $F_{\mathbb{C}}^n$. On V_1 of $\mathfrak{f}_{\mathbb{C}}^n$, we consider the inner product with e_1, ie_1, e_2, ie_2 as an orthonormal basis. This norm on V_1 then induces a Carnot metric d_c on $F_{\mathbb{C}}^n$.

Recall that, for a connected and simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. We shall identify $\mathfrak{g}$ and G via the exponential map and denote the group operation by $*$.

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