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Srinivas' Problem for Rational Double Points

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Abstract. For the completion B of a local geometric normal domain, V. Srinivas asked which subgroups of $\mathrm{Cl}\, B$ arise as the image of the map $\mathrm{Cl}\, A \to \mathrm{Cl}\, B$ on class groups as A varies among normal geometric domains with $B \cong \hat{A}$. For two-dimensional rational double point singularities, we show that all subgroups arise in this way by applying Noether–Lefschetz theory to linear systems with nonreduced base loci. By a similar technique we also show that in any dimension, every local ring of a normal hypersurface singularity has completion isomorphic to the completion of a geometric UFD.

1. Introduction

V. Srinivas posed several interesting problems about class groups of Noetherian local normal domains in his survey paper on geometric methods in commutative algebra [21, §3]. Recall that if A is such a ring with completion $\hat{A}$, then there is a well-defined injective map on divisor class groups $j : \mathrm{Cl}\, A \to \mathrm{Cl}\, \hat{A}$ [19, §1, Proposition 1] arising from valuation theory. For geometric local rings, that is, localizations of $\mathbb{C}$-algebras of finite type, Srinivas asks about the possible images of the map j [21, Questions 3.1 and 3.7].

Problem 1.1. Let B be the completion of a local geometric normal domain.

(a) What are the possible images of $\mathrm{Cl}\, A \hookrightarrow \mathrm{Cl}\, B$ as A ranges over all geometric local normal domains with $\hat{A} \cong B$?
(b) Is there a geometric normal local domain A with $\hat{A} \cong B$ and $\mathrm{Cl}\, A = \langle \omega_B \rangle \subset \mathrm{Cl}\, B$?

While we are mainly interested in (a), let us review the progress on Problem 1.1 (b). Since the dualizing module ω_B is necessarily in the image of $\mathrm{Cl}\, A \hookrightarrow \mathrm{Cl}\, B$ whenever A is a quotient of a regular local ring [15], part (b) asks whether the image that is a priori the *smallest* possible can be achieved. Moreover, if B is Gorenstein, then ω_B is trivial in $\mathrm{Cl}\, B$, and part (b) asks whether $\mathrm{Cl}\, A = 0$ is possible; in other words, whether B is the completion of a unique factorization domain (UFD). For arbitrary rings, Heitmann [9] proved that B is the completion of a UFD if and only if B is a field, a discrete valuation ring, or $\dim B \geq 2$, depth $B \geq 2$, and every integer is a unit in A, but for $\dim A \geq 2$, his constructions produce rings that are far from geometric.

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