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On Undulation Invariants of Plane Curves

A. Popolitov & Sh. Shakirov

Abstract. A classical problem introduced by A. Cayley and G. Salmon in 1852 is to determine if a given plane curve of degree $r > 3$ has undulation points, the points where the tangent line meets the curve with multiplicity four. They proved that there exists an invariant of degree $6(r-3)(3r-2)$ that vanishes if and only if the curve has undulation points. In this paper we give explicit formulas for this invariant in the case of quartics ($r = 4$) and quintics ($r = 5$), expressing it as the determinant of a matrix with polynomial entries, of sizes 21×21 and 36×36, respectively.

1. Introduction

This paper is devoted to a problem in classical invariant theory of plane curves, due to A. Cayley and G. Salmon (see [1], p. 362). Consider, on the projective plane $\mathbb{CP}^2$ with homogeneous coordinates $x_1 : x_2 : x_3$, a plane curve

$$P(x_1, x_2, x_3) = \sum_{i+j+k=r} C_{ijk} x_1^i x_2^j x_3^k = 0,$$

where P is a homogeneous irreducible degree r polynomial. By the Bezout theorem, any line in $\mathbb{CP}^2$ crosses this curve in exactly r points, if counted with multiplicities. The types of possible intersections thus can be put into correspondence with partitions $r = m_1 + m_2 + \cdots$, where the parts m_i of the partition are the multiplicities of intersection points. An illustration of this for the case of quartics is given on Figure 1.

If a line is generic, then it intersects the curve in r distinct points with all multiplicities 1, that is, it corresponds to the partition $(1, 1, \ldots, 1)$. The simplest nongeneric intersection occurs for the *tangent line* to a curve: then one of the intersection points has multiplicity 2 (the point of tangency), whereas all the other intersection points have multiplicity 1. This type of intersection corresponds to the partition $(2, 1, 1, \ldots)$. The next-to-simplest types of intersection are, respectively, $(3, 1, \ldots, 1)$ and $(2, 2, 1, \ldots, 1)$; the former situation is called a *line of inflection,* whereas the latter is called a *bitangent* since in this case the line is simultaneously tangent to a curve in two distinct points. One can continue further by considering lines of type $(4, 1, \ldots, 1)$, $(3, 2, 1, \ldots, 1)$, and so on. These generally do not have given names, with one notable exception: a line of type $(4, 1, \ldots, 1)$ is called a *line of undulation,* and the corresponding point of intersection is called an *undulation*

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