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Cycles of Polynomial Mappings in Several Variables over Discrete Valuation Rings and over Z

T. PEZDA

ABSTRACT. We find all possible cycle lengths of polynomial mappings in several variables over unramified discrete valuation domains. As a consequence, we determine the sets of all cycle lengths in R^N (where $N \geq 2$) for some Dedekind rings R. Finding these sets for $R = Z$ and any N is the main purpose of this paper.

1. Introduction

For a commutative ring R with unity and $\Phi = (\Phi_1, \dots, \Phi_N)$, where $\Phi_i \in R[X_1, \dots, X_N]$, we define a *cycle* for Φ as a k-tuple $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{k-1}$ of different elements of R^N such that

$$\Phi(\bar{x}_0) = \bar{x}_1, \qquad \Phi(\bar{x}_1) = \bar{x}_2, \qquad \dots, \qquad \Phi(\bar{x}_{k-1}) = \bar{x}_0.$$

The number k is called the *length* of this cycle.

Let $\mathcal{CYCL}(R, N)$ be the set of all possible cycle lengths for polynomial mappings in N variables with coefficients from R (we clearly assume that the elements of the considered cycles lie in R^N).

The main motivation to write this paper is finding $\mathcal{CYCL}(Z, N)$ for all natural N. As an exercise, one may treat the equality $\mathcal{CYCL}(Z, 1) = \{1, 2\}$. In [Pe2], the formula $\mathcal{CYCL}(Z, 2) = \{24, 18, 16, \text{and divisors}\}$ was established. In [Pe5], it was shown that the biggest element in $\mathcal{CYCL}(Z, N)$ equals $2 \cdot 4^N + o(4^N)$.

One of the main ingredients in obtaining these results is a local-to-global principle for polynomial cycles (see Section 2.4). This principle for $N \geq 2$ gives an expression of $\mathcal{CYCL}(R, N)$ in terms of $\mathcal{CYCL}(R_{\mathfrak{p}}, N)$, where $\mathfrak{p}$ runs over the family of all nonzero prime ideals of a Dedekind domain R.

Thus, in order to determine $\mathcal{CYCL}(Z, N)$, it is enough to determine $\mathcal{CYCL}(Z_p, N)$ for all prime p, where Z_p denotes the ring of p-adic numbers. In fact (see Theorem 2), it suffices to determine $\mathcal{CYCL}(Z_2, N)$ and $\mathcal{CYCL}(Z_3, N)$.

Using the notation of Theorem 1 and Section 2.1, we see that Z_p is a discrete valuation ring (DVR) of characteristic zero satisfying $e = 1$ (and therefore unramified). For the rings Z_p, the number f equals 1.

The main result of this paper is the following:

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