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Quasiconformal Homogeneity and Subgroups of the Mapping Class Group

Nicholas G. Vlamis

Abstract. In the vein of Bonfert-Taylor, Bridgeman, Canary, and Taylor we introduce the notion of quasiconformal homogeneity for closed oriented hyperbolic surfaces restricted to subgroups of the mapping class group. We find uniform lower bounds for the associated quasiconformal homogeneity constants across all closed hyperbolic surfaces in several cases, including the Torelli group, congruence subgroups, and pure cyclic subgroups. Further, we introduce a counting argument providing a possible path to exploring a uniform lower bound for the nonrestricted quasiconformal homogeneity constant across all closed hyperbolic surfaces.

1. Introduction

Let M be a hyperbolic manifold, and $\mathrm{QC}(M)$ be the associated group of quasiconformal homeomorphisms from M to itself. Given any subgroup $\Gamma \leq \mathrm{QC}(M)$, we say that M is Γ-*homogeneous* if the action of Γ on M is transitive. Furthermore, we say that M is Γ_K-*homogeneous* for $K \in [1, \infty)$ if the restriction of the action of Γ on M to the subset

$$\Gamma_K = \{f \in \Gamma : \ K_f \leq K\}$$

on M is transitive, where $K_f = \inf\{K : \ f \text{ is } K\text{-quasiconformal}\}$ is the *dilatation* of f.

If $\Gamma = \mathrm{QC}(M)$ and there exists a K such that M is Γ_K-homogeneous, then this manifold is said to be K-*quasiconformally homogeneous,* or K-*qch.* In [BTCMT05] it is shown that for each $n \geq 3$, there exists a constant $K_n > 1$ such that if $M \neq \mathbb{H}^n$ is an n-dimensional K-quasiconformally homogeneous hyperbolic manifold, then $K \geq K_n$. This result relies on rigidity in higher dimensions, which does not occur in dimension two. The natural question motivating this paper is as follows.

Question 1.1. *Does there exist a constant $K_2 > 1$ such that every K-qch surface $X \neq \mathbb{H}^2$ satisfies $K \geq K_2$?*

Let $\mathrm{Homeo}^+(S)$ be the group of orientation-preserving homeomorphisms of a surface S. Then the *mapping class group* of S is defined to be $\pi_0(\mathrm{Homeo}^+(S))$ and is denoted $\mathrm{Mod}(S)$. Given a closed hyperbolic surface X and $f \in \mathrm{QC}(X)$, let $[f] \in \mathrm{Mod}(X)$ denote its homotopy class. Then, the map $\pi : \mathrm{QC}(X) \to \mathrm{Mod}(X)$

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