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Merging Divisorial with Colored Fans

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Abstract. Given a spherical homogeneous space G/H of minimal rank, we provide a simple procedure to describe its embeddings as varieties with torus action in terms of divisorial fans. The torus in question is obtained as the identity component of the quotient group N/H, where N is the normalizer of H in G. The resulting Chow quotient is equal to (a blowup of) the simple toroidal compactification of $G/(HN^\circ)$. In the horospherical case, for example, it is equal to a flag variety, and the slices (coefficients) of the divisorial fan are merely shifts of the colored fan along the colors.

1. Introduction

We are working over the base field $\mathbb{C}$. Normal varieties X coming with an effective action of an algebraic torus $\mathbb{T}$, also called $\mathbb{T}$-varieties, can be encoded by divisorial fans $\mathcal{S}^X = \sum_{D\subseteq Y} \mathcal{S}^X_D \otimes D$ on algebraic varieties Y of dimension equal to the complexity of the torus action. In this notation, $D \subseteq Y$ runs through all prime divisors on Y, and $\mathcal{S}^X_D$ denotes a combinatorial object associated to D (being nontrivial for finitely many summands only). Let N denote the lattice of one-parameter subgroups of $\mathbb{T}$. Every $\mathcal{S}^X_D$ stands for a polyhedral subdivision of $N_\mathbb{Q}$ together with a prescribed labeling of its cells referring to the set of affine charts covering X.

The $\mathbb{T}$-variety X in question is then given as a contraction of a toric fibration over Y, and the data D and $\mathcal{S}^X_D$ describe exactly where and how this fibration degenerates, respectively. Vice versa, X can be reconstructed explicitly from $\mathcal{S}^X$ in two steps. First, one glues certain relative spectra over Y; the result of this procedure is called $\widetilde{\mathbb{TV}}(\mathcal{S}^X)$. Finally, we obtain X as $\mathbb{TV}(\mathcal{S}^X)$, which denotes a certain birational contraction of $\widetilde{\mathbb{TV}}(\mathcal{S}^X)$. See Section 2 for further details.

1.1. The Comparison Theorem

Let G be a connected reductive group, and $H \subset G$ a spherical subgroup such that the spherical homogenous space G/H is of *minimal rank* (see Definition 3.5). The

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