Michigan Math. J. 63 (2014), 865–888

Grothendieck Classes of Quiver Cycles as Iterated Residues

Justin Allman

Abstract. In the case of Dynkin quivers, we establish a formula for the Grothendieck class of a quiver cycle as the iterated residue of a certain rational function, for which we provide an explicit combinatorial construction. Moreover, we utilize a new definition of the double stable Grothendieck polynomials due to Rimányi and Szenes in terms of iterated residues to exhibit that the computation of quiver coefficients can be reduced to computing the coefficients in a combinatorially prescribed Laurent expansion of the aforementioned rational function.

1. Introduction

Let Q be a quiver with a finite vertex set $Q_0 = \{1, \ldots, N\}$ and finite set of arrows Q_1. For each arrow $a \in Q_1$, denote the vertex at its *head* by $h(a)$ and the vertex at its *tail* by $t(a)$. Throughout the sequel, we will refer also to the set

$$T(i) = \{j \in Q_0 \mid \exists a \in Q_1 \text{ with } h(a) = i \text{ and } t(a) = j\}. \tag{1}$$

Given a *dimension vector* of nonnegative integers $\mathbf{v} = (v_1, \ldots, v_N)$, define the vector spaces $E_i = \mathbb{C}^{v_i}$ and the affine *representation space* $V = \bigoplus_{a \in Q_1} \operatorname{Hom}(E_{t(a)}, E_{h(a)})$ with a natural action of the algebraic group $\mathbb{G} = GL(E_1) \times \cdots \times GL(E_N)$ given by

$$(g_i)_{i \in Q_0} \cdot (f_a)_{a \in Q_1} = (g_{h(a)} f_a g_{t(a)}^{-1})_{a \in Q_1}. \tag{2}$$

A *quiver cycle* $\Omega \subset V$ is a $\mathbb{G}$-stable, closed, irreducible subvariety and, as such, has a well-defined structure sheaf $\mathcal{O}_\Omega$. The goal of this paper is the calculation of the class

$$[\mathcal{O}_\Omega] \in K_{\mathbb{G}}(V),$$

in the $\mathbb{G}$-equivariant Grothendieck ring of V. To accomplish this, we reformulate the problem in an equivalent setting; we realize $[\mathcal{O}_\Omega]$ as the K-class associated to a certain degeneracy locus of a quiver of vector bundles over a smooth complex projective base variety X.

Formulas for this class exist already in the literature, the most general of which is due to Buch [Buc08], and which we now explain. Buch's result is given in terms of the stable version of Grothendieck polynomials first invented by Lascoux and Schützenberger [LS82] as representatives of structure sheaves of Schubert varieties in a flag manifold, which are applied to the E_i in an appropriate way. For details specific to this context, see [Buc08], Section 3.2, and for a comprehensive

Received December 20, 2013. Revision received August 20, 2014.