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Four-Dimensional Compact Manifolds with Nonnegative Biorthogonal Curvature

Ezio Costa & Ernani Ribeiro Jr.

Abstract. The goal of this article is to study the pinching problem proposed by S.-T. Yau in 1990 replacing sectional curvature by a weaker condition on biorthogonal curvature. Moreover, we classify four-dimensional compact oriented Riemannian manifolds with nonnegative biorthogonal curvature. In particular, we obtain a partial answer to the Yau conjecture on pinching theorem for four-dimensional compact manifolds.

1. Introduction

In the last century, very much attention has been given to four-dimensional compact Riemannian manifolds with positive scalar curvature. A classical problem in geometry is to classify such manifolds in the category of either topology, diffeomorphism, or isometry. This subject have been studied extensively because of their connections with general relativity and quantum theory. For comprehensive references on such a theory, we indicate, for instance, [1; 3; 5; 7; 12; 15; 17; 20], and [22]. Arguably, classifying four-dimensional compact Riemannian manifolds or understanding their geometry is definitely an important issue.

In 1990, S.-T. Yau collected some important open problems. Here, we call attention to the paragraph where he wrote:

> *"The famous pinching problem says that on a compact simply connected manifold if $K_{\min} > \frac{1}{4}K_{\max} > 0$, then the manifold is homeomorphic to a sphere. If we replace $K_{\max}$ by normalized scalar curvature, can we deduce similar pinching results?"* (See [24], problem 12, page 369; see also [27].)

In other words, Yau's conjecture on pinching theorem can be rewritten as follows (see [13]).

Conjecture 1 (Yau, 1990). *Let (M^n, g) be a compact simply connected Riemannian manifold. Denote by s_0 the normalized scalar curvature of M^n. If $K_{\min} > \frac{n-1}{n+2}s_0$, then M^n is diffeomorphic to a standard sphere $\mathbb{S}^n$.*

A classical example obtained in [13] shows that $\frac{n-1}{n+2}$ is the best possible pinching for this conjecture (see Example 3.1 in [13]). We also notice that if s is the scalar curvature of a Riemannian manifold M^n, then the normalized scalar curvature of M^n is given by $s_0 = \frac{s}{n(n-1)}$.

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