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Singular Rationally Connected Surfaces with Nonzero Pluri-Forms

Wenhao Ou

Abstract. Let X be a projective rationally connected surface with canonical singularities carrying a nonzero reflexive pluri-form, that is, the reflexive hull of $(\Omega_X^1)^{\otimes m}$ has a nonzero global section for some positive integer m. We show that any such surface X can be obtained from a rational ruled surface by a very explicit sequence of blow-ups and blow-downs. Moreover, we interpret the existence of nonzero pluri-forms in terms of semistable reduction.

1. Introduction

Recall that a projective variety X is said to be rationally connected if for any two general points in X, there exists a rational curve passing through them; see [Kol96, Def. 3.2 and Prop. 3.6]. It is known that for a smooth projective rationally connected variety X, $H^0(X, (\Omega_X^1)^{\otimes m}) = \{0\}$ for $m > 0$; see [Kol96, Cor. IV.3.8]. In [GKKP11, Thm. 5.1], it is shown that if a pair (X, D) is klt and X is rationally connected, then $H^0(X, \Omega_X^{[m]}) = \{0\}$ for $m > 0$, where $\Omega_X^{[m]}$ is the reflexive hull of Ω_X^m. By [GKP12, Thm. 3.3], if X is factorial, rationally connected and has canonical singularities, then $H^0(X, (\Omega_X^1)^{[\otimes m]}) = \{0\}$ for $m > 0$, where $(\Omega_X^1)^{[\otimes m]}$ is the reflexive hull of $(\Omega_X^1)^{\otimes m}$. However, this is not true without the assumption of being factorial; see [GKP12, Example 3.7]. In this paper, our aim is to classify rationally connected surfaces with canonical singularities that have nonzero reflexive pluri-forms. We will give two methods to construct such surfaces (see Construction 1.2 and Construction 1.6), and we will also prove that every such surface can be constructed by both of these methods (see Theorem 1.3 and Theorem 1.5). This gives an affirmative answer to [GKP12, Remark and Question 3.8]

The following example is given in [GKP12, Example 3.7].

Example 1.1. Let $\pi' : X' \to \mathbb{P}^1$ be any smooth ruled surface. Choose four distinct points q_1, q_2, q_3, q_4 in $\mathbb{P}^1$. For each point q_i, perform the following sequence of birational transformations of the ruled surface:

(i) Blow up a point x_i in the fiber over q_i. Then we get two (-1)-curves that meet transversely at x_i'.

(ii) Blow up the point x_i'. Over q_i, we get two disjoint (-2)-curves and one (-1)-curve. The (-1)-curve appears in the fiber with multiplicity two.

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