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Dilatation, Pointwise Lipschitz Constants, and Condition N on Curves

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Abstract. Let $f : X \to Y$ be a homeomorphism between locally compact Ahlfors Q-regular metric spaces, $Q > 1$. We prove that finiteness of either $\mathrm{lip}_f(x)$, $h_f(x)$, or $h_f^*(x)$ for every $x \in X \setminus E$ implies that f satisfies Lusin's condition N on p-almost every curve (in the sense of curve modulus), provided that the exceptional set E has σ-finite Hausdorff $(n-p)$-measure. Here $h_f(x)$ and h_f^* are the linear dilatations of f and f^{-1} at x, and $\mathrm{lip}_f(x)$ is the pointwise Lipschitz constant (each defined with lim inf rather than lim sup).

As a corollary, we improve a theorem of Balogh, Koskela, and Rogovin on the Sobolev regularity of mappings of finite and essentially bounded dilatation.

Furthermore, we show that for nonhomeomorphic continuous mappings into arbitrary targets, finiteness of $\mathrm{lip}_f(x)$ away from E still implies condition N on p-almost every curve.

1. Introduction

Let $f : X \to Y$ be a homeomorphism, where $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \nu)$ are locally Ahlfors Q-regular metric measure spaces X and Y, with $Q > 1$. Recall that local Q-regularity merely means that the measure of a small ball of radius r is comparable to r^Q (see Section 2 for precise definitions).

We are primarily interested in the pointwise constants

$$\mathrm{lip}_f(x) := \liminf_{r\to 0} \frac{L_f(x,r)}{r} \quad \text{and} \quad h_f(x) := \liminf_{r\to 0} h_f(x,r),$$

where

$$L_f(x,r) = \sup_{x'\in B(x,r)} d(f(x'), f(x)), \qquad l_f(x,r) = \inf_{x'\in X\setminus B(x,r)} d(f(x'), f(x)),$$

$$h_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)}.$$

We also let $h_f^*(x) = h_{f^{-1}}(f(x))$. One can define Lip_f, H_f, and H_f^* similarly, with lim sup replacing lim inf, but we shall mostly be interested only in the lim inf case.

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