Michigan Math. J. 63 (2014), 553–569

Counting Genus One Fibered Knots in Lens Spaces

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Abstract. The braid axis of a closed 3-braid lifts to a genus one fibered knot in the double cover of S^3 branched over the closed braid. Every genus one fibered knot in a 3-manifold may be obtained in this way. Using this perspective, we answer a question of Morimoto about the number of genus one fibered knots in lens spaces. We determine the number of genus one fibered knots up to homeomorphism and up to isotopy in any given lens space. This number is 3 in the case of the lens space $L(4, 1)$, 2 for the lens spaces $L(m, 1)$ with $m > 0$ and $m \neq 4$, and at most 1 otherwise. Furthermore, each homeomorphism equivalence class in a lens space is realized by at most two isotopy classes.

1. Introduction

Let M be a 3-manifold. We say that a knot K in M is a *genus one fibered knot,* GOF-knot for short, if $M - N(K)$ is a once-punctured torus bundle over the circle and the boundary of a fiber is a longitude of K. In particular, we will always consider a GOF-knot to be null homologous.

As begun by Burde and Zieschang in [BZ67], González-Acuña [GAn70] shows that the trefoil (and its mirror) and the figure-eight knot are the only GOF-knots in S^3. Morimoto shows that up to homeomorphism each lens space $L(m, 1)$ contains at least two GOF-knots if $m > 0$ and exactly two if $m \in \{1, 2, 3, 5, 19\}$, $L(4, 1)$ contains exactly three GOF-knots, each of $L(0, 1)$, $L(5, 2)$, and $L(19, 3)$ contains exactly one GOF-knot, and each of $L(19, 2)$, $L(19, 4)$, and $L(19, 7)$ contains no GOF-knots, [Mor89]. Morimoto then asks the following question.

Question [Mor89]. Are the numbers of GOF-knots in all lens spaces bounded?

In this article we use double branched covers of two-bridge links represented as closed 3-braids to address this question.

Theorem 4.3. *Up to homeomorphisms, the lens space $L(\alpha', \beta')$ contains exactly*

- *three distinct GOF-knots if and only if $L(\alpha', \beta') \cong L(4, 1)$,*
- *two distinct GOF-knots if and only if $L(\alpha', \beta') \cong L(\alpha, 1)$ for $\alpha > 0$ and $\alpha \neq 4$,*

Received August 20, 2013. Revision received May 5, 2014.

This work was partially supported by a VIGRE postdoc under NSF grant number DMS-0089927 to the University of Georgia at Athens, by NSF grant number DMS-0707509, and by Simons Foundation grant #209184 to Kenneth L. Baker.