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Exotic Blowup Solutions for the u^5 Focusing Wave Equation in $\mathbb{R}^3$

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Abstract. For the critical focusing wave equation $\Box u = u^5$ on $\mathbb{R}^{3+1}$ in the radial case, we construct a family of blowup solutions that are obtained from the stationary solutions $W(r)$ by means of a dynamical rescaling $\lambda(t)^{1/2}W(\lambda(t)r)+\texttt{correction}$ with $\lambda(t) \to \infty$ as $t \to 0$. The novelty here lies with the scaling law $\lambda(t)$ that eternally oscillates between various pure-power laws.

1. Introduction

The energy critical focusing wave equation in $\mathbb{R}^3$

$$\Box u = u^5, \quad \Box = \partial_t^2 - \triangle \tag{1.1}$$

has been the subject of intense investigations in recent years. This equation is known to be locally well posed in the space $\mathcal{H} := \dot{H}^1 \times L^2(\mathbb{R}^3)$, meaning that if $(u(0), u_t(0)) \in \mathcal{H}$, then there exists a solution locally in time and continuous in time taking values in $\mathcal{H}$. Solutions need to be interpreted in the Duhamel sense:

$$u(t) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}u^5(s)\,ds. \tag{1.2}$$

These solutions have finite energy:

$$E(u, u_t) = \int_{\mathbb{R}^3}\left[\frac{1}{2}(u_t^2 + |\nabla u|^2) - \frac{u^6}{6}\right]dx = \text{ const.}$$

The remarkable series of papers [2; 3; 4; 5] establishes a complete classification of all *possible* type-II blow up dynamics in the radial case. It remains, however, to investigate the *existence* of *all* allowed scenarios in this classification. Steps in this direction were undertaken in [1; 8; 11], where a constructive approach to actually exhibit and thereby prove the existence of such type-II dynamics was undertaken. Recall that a type-II blow up solution $u(t, x)$ with blowup time T_* is one for which

$$\limsup_{t\to T_*} \|u(t,\cdot)\|_{\dot{H}^1} + \|u_t(t,\cdot)\|_{L^2} < \infty$$

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