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Smoothly Slice Boundary Links Whose Derivative Links Have Nonvanishing Milnor Invariants

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Abstract. We give an example of a 3-component smoothly slice boundary link, each of whose components has a genus one Seifert surface, such that any metaboliser of the boundary link Seifert form is represented by three curves on the Seifert surfaces that form a link with nonvanishing Milnor triple linking number. We also give a generalization to m-component links and higher Milnor invariants. We prove that our examples are ribbon and that all ribbon links are boundary slice.

1. Introduction

The topological four-dimensional surgery conjecture for free groups states that the surgery sequence discussed in [FQ90, Section 11.3] is exact when the fundamental group is free. A key test case is the question of whether the Whitehead double of the Borromean rings is a topologically slice link [Fre84; CF84; Fre93; Kru08]. One strategy to slice a boundary link $L = L_1 \sqcup \cdots \sqcup L_m$ (a boundary link is a link whose components bound disjoint Seifert surfaces in S^3) is to push these Seifert surfaces $F = F_1 \sqcup \cdots \sqcup F_m$, $\partial F_i = L_i$, into the 4-ball B^4 and then to ambiently surger the Seifert surfaces to discs by finding a set of curves generating a half-rank submodule of $H_1(F_i;\mathbb{Z})$ for each $i = 1, \dots, m$ and finding framed discs, pairwise disjoint, embedded in $B^4 \smallsetminus F$, and with boundary these curves.

In order for such framed discs to exist, such a set of curves must be a *metaboliser* for the boundary link Seifert form (see Definitions 2.4 and 2.5). Following [CHL10], we consider simple closed curves on the Seifert surfaces representing a metaboliser M as a link in S^3, also denoted by M, and call this the *derivative* of L with respect to the metaboliser.

If a derivative is itself a slice link, then the programme works, and the original link is slice. On the other hand, if we have a boundary link with Seifert surfaces and we know that all of the metabolisers are not slice links, we can wonder if this implies that the link is not slice. In the famous case of the Whitehead double of the Borromean rings, with their obvious Seifert surfaces, all derivatives are the Borromean rings, which are well known not to be slice; for example, they have a nonzero Milnor triple linking number $\overline{\mu}(123)$. Also, the Whitehead double of the

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