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When Are Two Coxeter Orbifolds Diffeomorphic?

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Abstract. One can define what it means for a compact manifold with corners to be a "contractible manifold with contractible faces." Two combinatorially equivalent, contractible manifolds with contractible faces are diffeomorphic if and only if their four-dimensional faces are diffeomorphic. It follows that two simple convex polytopes are combinatorially equivalent if and only if they are diffeomorphic as manifolds with corners. On the other hand, by a result of Akbulut, for each $n \geq 4$, there are smooth, contractible n-manifolds with contractible faces that are combinatorially equivalent but not diffeomorphic. Applications are given to rigidity questions for reflection groups and smooth torus actions.

1. Introduction

More than once during the past few years I have been asked, "Are combinatorially equivalent Coxeter orbifolds diffeomorphic?" (Taras Panov and Mikiya Masuda asked me this at a 2010 conference on toric topology in Banff. More recently, Suhyoung Choi asked me the same question in connection with projective representations of Coxeter groups, cf. [5, Question 3.3].) By "Coxeter orbifold" the questioner means something like the fundamental polytope of a geometric reflection group. The underlying space of such an orbifold has the structure of a manifold with corners. Usually, the questioner also wants to require that the underlying space is diffeomorphic, as a manifold with corners, to a simple convex polytope. In this context the answer to the question is affirmative although the proof is not obvious (see Wiemeler [28, Corollary 5.3] and Corollary 1.3). However, if one weakens the definition by only requiring the strata to be compact contractible manifolds, there are counterexamples (cf. Theorem 1.4). The problem is caused by the four-dimensional strata.

An orbifold Q is *reflection type* if its local models are finite linear reflection groups on $\mathbb{R}^n$. (These were called "reflectofolds" in [11].) Since the orbit space of a finite linear reflection group is the product of a Euclidean space with a simplicial cone, a smooth orbifold of reflection type naturally has the structure of a smooth manifold with corners. One can label each codimension two stratum by an integer $m \geq 2$ to indicate that the local dihedral group along the stratum has order $2m$. So, Q is a smooth manifold with corners, and the labeling is such that it determines a *finite* Coxeter group of rank k on each codimension k stratum. Conversely, if Q

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