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On Volumes of Complex Hyperbolic Orbifolds

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Abstract. We construct an explicit lower bound for the volume of a complex hyperbolic orbifold that depends only on dimension.

0. Introduction

A *hyperbolic orbifold* is a quotient of real, complex, quaternionic or octonionic hyperbolic space by a discrete group of isometries, usually denoted by Γ. An orbifold is a *manifold* when Γ contains no elements of finite order.

The real hyperbolic 2-orbifold of minimum volume was identified by Siegal [20]. An analogous result in dimension three was proved by Gehring and Martin [7]. In the remaining dimensions and algebras of definition, the existence of a hyperbolic orbifold of minimum volume is guaranteed by a theorem of Wang [22].

In this paper, we prove an explicit lower bound for the volume of any complex hyperbolic orbifold that depends only on dimension. Our methods here are similar to those of the prequel [2], which addressed the real hyperbolic case. The complex setting provides an additional corollary, and the corresponding Lie group curvature calculations are of independent interest.

Let $\mathbf{H}^n_{\mathbb{C}}$ denote complex hyperbolic n-space. The holomorphic sectional curvature of $\mathbf{H}^n_{\mathbb{C}}$ is normalized to be -1; accordingly, the sectional curvatures are pinched between -1 and $-1/4$. Let $\mathrm{SU}(n,1)$ denote the indefinite special unitary group of indicated signature. This group is a Lie group, and it acts transitively by isometries on complex hyperbolic space. With an appropriate scale of a canonical metric on $\mathrm{SU}(n,1)$, we define a *Riemannian submersion* $\pi : \mathrm{SU}(n,1)/\Gamma \to \mathbf{H}^n_{\mathbb{C}}/\Gamma$. The volume of a complex hyperbolic n-orbifold is thereby described in terms of the volume of the fundamental domain of a lattice in a Lie group. The latter is then bounded from below using results of Wang [22] and Gunther (see e.g. [6]). In what follows, dimension will refer to complex dimension, unless otherwise stated.

Theorem 0.1. *The volume of a complex hyperbolic n-orbifold is bounded below by $\mathcal{C}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{C}(n) = \frac{2^{n^2+n+1}\pi^{n/2}(n-1)!\,(n-2)!\cdots!\,3!\,2!\,1!}{(36n+21)^{(n^2+2n)/2}\Gamma((n^2+2n)/2)} \times \int_0^{\min[0.06925\sqrt{36n+21},\pi]} \sin^{n^2+2n-1}\rho\, d\rho.$$

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