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Flat Bundles and Commutator Lengths

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1. Introduction

The study of commutator lengths in various structure groups for fiber bundles has a long history in topology, where Milnor's 1958 paper on flat disk bundles [11] and Wood's follow-up 1971 paper on flat circle bundles [16] played avantgarde roles. More recently, several results on (stable) commutator lengths in mapping class groups of surfaces were obtained with the help of gauge theory; see, for instance, [5; 8; 9]. In [2], Korkmaz, Monden, and the author proved the following theorem in the same vein: Let δ be a boundary parallel simple-closed curve on an orientable surface Σ of genus $g \geq 2$ with boundary, and let t_δ denote the positive Dehn twist along δ in the mapping class group $\mathrm{Map}(\Sigma)$. Then the commutator length of t_δ^n is $\lfloor \frac{|n|+3}{2} \rfloor$, the floor of $\frac{|n|+3}{2} \in \mathbb{Z}[\frac{1}{2}]$. This led the first precise calculation of a nonzero stable commutator length of an element in a mapping class group of a surface of genus $g \geq 2$. (See Theorem 5 below.) The authors' proof of this theorem relied on celebrated results on Seiberg–Witten invariants of symplectic 4-manifolds and on mapping class group factorizations featuring a generalized lantern relation. We will give a more elementary proof of this theorem by reviewing an argument of Morita in [12] using Euler classes of fiber bundles and the Milnor–Wood inequalities on one hand and by employing push maps on the other; see Section 2.

A curious open question on surface bundles asks whether or not there exists a closed surface bundle over a closed surface that does not admit a flat connection. An "approximate" answer to this problem was given by Bestvina, Church, and Souto [3], who proved that the Atiyah–Kodaira surface bundles do not admit flat connections for which some distinguished sections are flat. The Atiyah–Kodaira examples are holomorphic bundles on complex surfaces of general type, whose construction for given fiber and base genera (g, h) (with the desired sections of high self-intersection numbers) is a rather challenging task. Moreover, Parshin's proof of the geometric Shafarevich conjecture implies that there can be at most finitely many of such examples for fixed (g, h). However, these bundles being holomorphic is irrelevant to the above question – as we will see, it can be dropped so as to obtain a much more general result, which is the main theorem of this article:

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