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Upper Bounds for the Minimal Number of Singular Fibers in a Lefschetz Fibration over the Torus

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Abstract. In this paper, we give some relations in the mapping class groups of oriented closed surfaces in the form that a product of a small number of right-hand Dehn twists is equal to a single commutator. Consequently, we find upper bounds for the minimal number of singular fibers in a Lefschetz fibration over the torus.

1. Introduction

Lefschetz fibrations were originally introduced for studying topological properties of smooth complex projective varieties and afterwards generalized to differentiable category. Furthermore Donaldson and Gompf revealed the close relationship between Lefschetz fibrations and 4-dimensional symplectic topology in the late 1990s, and since then they have been extensively studied.

The information about the number of singular fibers in a Lefschetz fibration provides us important information about the topological invariants of its total space such as the Euler number, the signature, the Chern numbers, and so on. In addition, it has been known that the number of singular fibers in a Lefschetz fibration cannot be arbitrary, so it makes sense to ask what the minimal number of singular fibers in a Lefschetz fibration is. We denote by $N(g,h)$ the minimal number of singular fibers in a nontrivial relatively minimal genus g Lefschetz fibration over the oriented closed surface of genus h. This minimal number has been studied by various authors. Table 1 shows previous studies about $N(g,h)$. Korkmaz and Ozbagci [8] proved that (1) $N(g,h)=1$ if and only if $g \geq 3$ and $h \geq 2$, (2) $N(1,h)=12$ for all $h \geq 0$, and (3) $5 \leq N(2,h) \leq 8$ for all $h \geq 0$. The upper bound for $N(2,h)$ in (3) follows from the existence of a genus 2 Lefschetz fibration over the sphere with eight singular fibers, which was constructed by Matsumoto [11]. In addition, for $g=2$, Korkmaz and Stipsicz [10] showed that $N(2,h)=5$ for $h \geq 6$, and furthermore Monden [12] improved their results by showing that (1) $N(2,h)=5$ for all $h \geq 3$, (2) $N(2,2) \leq 6$, and (3) $6 \leq N(2,1) \leq 7$. Ozbagci [13] proved that the number of singular fibers in a genus 2 Lefschetz fibration over the sphere cannot be equal to 5 or 6, and Xiao [15] constructed a genus 2 Lefschetz fibration over the sphere with seven singular fibers; hence, $N(2,0)=7$. For $h=0$, some estimates for $N(g,0)$ are known. Cadavid [1] and Korkmaz [6] independently generalized Matsumoto's genus 2 Lefschetz fibration as above to genus g Lefschetz fibrations over the sphere with $2g+10$ singular fibers for g odd or with $2g+4$ singular fibers for g even. This

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