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A Geometric Criterion to Be Pseudo-Anosov

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Abstract. If S is a hyperbolic surface and $\mathring{S}$ the surface obtained from S by removing a point, the mapping class groups $\mathrm{Mod}(S)$ and $\mathrm{Mod}(\mathring{S})$ fit into a short exact sequence

$$1 \to \pi_1(S) \to \mathrm{Mod}(\mathring{S}) \to \mathrm{Mod}(S) \to 1.$$

We give a new criterion for mapping classes in the kernel to be pseudo-Anosov using the geometry of hyperbolic 3-manifolds. Namely, we show that if M is an ε-thick hyperbolic manifold homeomorphic to $S \times \mathbb{R}$, then an element of $\pi_1(M) \cong \pi_1(S)$ represents a pseudo-Anosov element of $\mathrm{Mod}(\mathring{S})$ if its geodesic representative is "wide." We establish similar criteria where M is replaced with a coarsely hyperbolic surface bundle coming from a δ-hyperbolic surface–group extension.

1. Introduction: Mapping Classes from Fibrations

If X is a surface, let $\mathrm{Mod}(X) = \pi_0(\mathrm{Homeo}^+(X))$ be its mapping class group, and let $\mathring{X}$ be the surface obtained from X by removing a point.

Surface bundles $X \to E \to B$ over a space B with fiber X are determined by homomorphisms $\pi_1(B) \to \mathrm{Mod}(X)$; see [21]. Thurston's geometrization theorem for fibered 3-manifolds opens the door to an investigation of the geometric behavior of such surface bundles. For instance, there are necessary and sufficient geometric conditions on $\pi_1(B) \to \mathrm{Mod}(X)$ that guarantee that $\pi_1(E)$ is word-hyperbolic; see [9; 10]. To verify these conditions, one is often faced with the problem of determining when a subgroup $G < \mathrm{Mod}(X)$ is purely pseudo-Anosov, a problem we take up here.

To describe our first result, let N be a closed hyperbolic 3-manifold that fibers over the circle with fiber a surface S, and let $N_{\mathbb{Z}} \to N$ be the corresponding infinite cyclic covering of N. The long exact sequence of the fibration is concentrated in a short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z} \longrightarrow 1, \tag{1.1}$$

which injects into the Birman exact sequence [4]

$$1 \longrightarrow \pi_1(S) \longrightarrow \mathrm{Mod}(\mathring{S}) \longrightarrow \mathrm{Mod}(S) \longrightarrow 1.$$

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