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On the Operator Λ^* and the Beurling–Ahlfors Transform on Radial Functions

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Abstract. We study the operator Λ^* that arises in the study of the action of the Beurling–Ahlfors transform on the class of radial functions. Using a novel estimate for pure-jump martingales, we provide a new proof of the weak-type $(1,1)$ estimate for Λ^*, originally established by J. Gill by completely different techniques.

1. Introduction

The purpose of this paper is to study an operator closely related to the Beurling–Ahlfors transform on the complex plane. We recall that the latter is the singular integral operator acting on $L^p(\mathbb{C})$ and defined by the formula

$$Bf(z) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,\mathrm{d}w,$$

where p.v. means the principal value, and the integration is with respect to the Lebesgue measure on the complex plane $\mathbb{C}$. This operator plays a fundamental role in the study of quasi-conformal mappings and partial differential equations (see [1] and references therein for an overview and applications). An important and interesting problem concerns the precise values of the L^p norms of this operator. This question has gained considerable interest in the literature, and the long standing conjecture of Iwaniec [10] states that

$$\|B\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} = p^* - 1,$$

where $p^* = \max\{p, p/(p-1)\}$. Whereas the lower bound of p^*-1 was obtained by Lehto [11], the question about the upper bound remains open. Thus far, the best results in this direction is the inequality $\|B\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \le 1.575(p^*-1)$, established in [2], and the bound $\|B\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \le 1.4(p^*-1)$ for $p \ge 1{,}000$, proved in [5]. Both these statements were shown by obtaining a martingale representation of the operator B and applying the probabilistic techniques of Burkholder [6; 7].

As a Calderón–Zygmund singular integral, the Beurling–Ahlfors operator is also of weak type $(1,1)$; see [14]. That is, it maps $L^1(\mathbb{C})$ into weak-$L^1(\mathbb{C})$. A problem of interest also is to determine the best constant in the weak-type $(1,1)$ inequality. Bañuelos and Janakiraman [3] studied the action of B on the space of

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