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On Some Hermitian Variations of Hodge Structure of Calabi–Yau Type with Real Multiplication

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Introduction

A Hodge structure of *Calabi–Yau or CY type* is an effective weight n Hodge structure with $h^{n,0} = 1$. The work of Gross [Gro94] and Sheng–Zuo [SZ10] shows that every Hermitian symmetric domain $\mathcal{D}$ carries a canonical $\mathbb{R}$-variation of Hodge structure (VHS) $\mathscr{V}$ of CY type (cf. also [FL11, §2] for more discussion). Furthermore, every other equivariant $\mathbb{R}$-VHS (or *Hermitian VHS*) of CY type on $\mathcal{D}$ is obtained from $\mathscr{V}$ using certain standard constructions (see [FL11, Theorem 2.22]). For example, each of the four rank 3 Hermitian symmetric tube domains $\mathcal{D}$, namely III_3, $\mathrm{I}_{3,3}$, II_6, and EVII (corresponding to the real Lie groups $\mathrm{Sp}(6,\mathbb{R})$, $\mathrm{SU}(3,3)$, $\mathrm{SO}^*(12)$, and $\mathrm{E}_{7,3}$ respectively), carries a weight 3 $\mathbb{R}$-VHS of CY type with the relevant Hodge number $h^{2,1} = 6$, 9, 15, and 27 respectively, and every primitive irreducible weight 3 Hermitian VHS of CY type that is also of tube type is of this form. Here primitive means that the VHS is not induced from a lower weight VHS in an obvious sense, and tube type means that the corresponding complex VHS is irreducible. This gives a satisfactory classification (over $\mathbb{R}$) of Hermitian VHS of CY type analogous to the classification of Satake [Sat65] and Deligne [Del79] of totally geodesic holomorphic embeddings of Hermitian symmetric domains into the Siegel upper half-space $\mathfrak{H}_g$, or equivalently Hermitian VHS of abelian variety type.

The analogous classification over $\mathbb{Q}$ of Hermitian VHS $\mathscr{V}$ of Calabi–Yau type is much more difficult. The weight 2 case, or *K3 type,* was analyzed by Zarhin [Zar83] and van Geemen [vG08]. A basic invariant measuring the difference between the classification over $\mathbb{Q}$ and over $\mathbb{R}$ is the algebra $E := \mathrm{End}_{\mathrm{Hg}}(V_s)$ of Hodge endomorphisms of a general fiber V_s of $\mathscr{V}$. In the Calabi–Yau type case, E is either a totally real field or a CM field (see [Zar83] or [FL11, Prop. 3.1]). If $E = E_0$ is a totally real field, we say that the Hermitian VHS has *weak real multiplication by* E_0. In the weight 3 case, we showed in [FL11, Theorem 3.18] that there are at most two primitive cases of Hermitian VHS of CY threefold type defined over $\mathbb{Q}$ with nontrivial weak real multiplication. These cases correspond to the domains $\mathrm{I}_{3,3}$ and II_6 associated to the groups $\mathrm{SU}(3,3)$ and $\mathrm{SO}^*(12)$ respectively. In the two other tube domain cases mentioned above, III_3 and EVII, nontrivial weak real multiplication cannot arise. For the $\mathrm{SU}(3,3)$ case, we showed

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