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# Rank Gradients of Infinite Cyclic Covers of 3-Manifolds

Jason DeBlois, Stefan Friedl, & Stefano Vidussi

Abstract. Given a 3-manifold $M$ with no spherical boundary components, and a primitive class $\phi \in H^1(M;\mathbb{Z})$, we show that the following are equivalent:
(1) $\phi$ is a fibered class,
(2) the rank gradient of $(M,\phi)$ is zero,
(3) the Heegaard gradient of $(M,\phi)$ is zero.

## 1. Introduction

A *directed 3-manifold* is a pair $(M,\phi)$ where $M$ is a compact, orientable, connected 3-manifold with toroidal or empty boundary, and $\phi \in H^1(M;\mathbb{Z}) = \operatorname{Hom}(\pi_1(M),\mathbb{Z})$ is a primitive class, that is, $\phi$ viewed as a homomorphism $\pi_1(M) \to \mathbb{Z}$ is an epimorphism. We say that a directed 3-manifold $(M,\phi)$ *fibers over* $S^1$ if there exists a fibration $p : M \to S^1$ such that the induced map $p_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ coincides with $\phi$. We refer to such $\phi$ as a *fibered class.*

It is well known that the pair $(\pi_1(M), \phi : \pi_1(M) \to \mathbb{Z})$ determines whether $\phi$ is fibered or not. Indeed, it follows from Stallings' theorem [St62] (together with the resolution of the Poincaré conjecture) that $\phi$ is a fibered class if and only if $\operatorname{Ker}(\phi : \pi_1(M) \to \mathbb{Z})$ is finitely generated.

Stallings' theorem can be generalized in various directions (see e.g. [FV12, Theorem 5.2], [SW09a; SW09b], and [FSW13]). Our main result gives a new fibering criterion, which is also a strengthening of Stallings' theorem. In order to state our result, we need the notion of rank gradient, which was first introduced by Lackenby [La05]. Given a finitely generated group $\pi$, we denote by $\operatorname{rk}(\pi)$ the *rank* of $\pi$, that is, the minimal number of generators of $\pi$. If $(M,\phi)$ is a directed 3-manifold, then we write

$$\pi_n = \operatorname{Ker}(\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_n),$$

and we refer to

$$\operatorname{rg}(M,\phi) := \liminf_{n\to\infty} \frac{1}{n} \operatorname{rk}(\pi_n)$$

as the *rank gradient* of $(M,\phi)$. (In the notation of [La05] this is the rank gradient of $(\pi_1 M, \{\pi_n\})$.)

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