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Examples of Dynamical Degree Equals Arithmetic Degree

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Introduction

Let $X/\mathbb{C}$ be a smooth projective variety, let $f : X \to X$ be a dominant rational map, and let $f^* : \mathrm{NS}(X)_{\mathbb{R}} \to \mathrm{NS}(X)_{\mathbb{R}}$ be the induced map on the Néron–Severi group $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes \mathbb{R}$. Further, let $\rho(T, V)$ denote the spectral radius of a linear transformation $T : V \to V$ of a real or complex vector space. Then the (*first*) *dynamical degree of* f is the quantity

$$\delta_f = \lim_{n\to\infty} \rho((f^n)^*, \mathrm{NS}(X)_{\mathbb{R}})^{1/n}.$$

Alternatively, if we let H be any ample divisor on X and $N = \dim(X)$, then δ_f is also given by the formula

$$\delta_f = \lim_{n\to\infty} ((f^n)^* H \cdot H^{N-1})^{1/n}.$$

See [13, Proposition 1.2(iii)] and [19]. Dynamical degrees have been much studied over the past couple of decades; see [19] for a partial list of references.

In two earlier papers [19; 26], the authors studied an analogous arithmetic degree, which we now describe. Assume that X and f are defined over $\bar{\mathbb{Q}}$, and write $X(\bar{\mathbb{Q}})_f$ for the set of points P whose forward f-orbit

$$\mathcal{O}_f(P) = \{P, f(P), f^2(P), \dots\}$$

is well defined. (There are always many such points; see [1].) Further, let

$$h_X : X(\bar{\mathbb{Q}}) \to [0, \infty)$$

be a Weil height on X relative to an ample divisor, and let $h_X^+ = \max\{1, h_X\}$. The *arithmetic degree of* f *at* $P \in X(\bar{\mathbb{Q}})_f$ is the quantity

$$\alpha_f(P) = \lim_{n\to\infty} h_X^+(f^n(P))^{1/n}, \tag{1}$$

assuming that the limit exists. We also define *upper and lower arithmetic degrees* by the formulas

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_X^+(f^n(P))^{1/n} \quad \text{and} \quad \underline{\alpha}_f(P) = \liminf_{n\to\infty} h_X^+(f^n(P))^{1/n}.$$

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