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The Exactness of a General Skoda Complex

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Abstract. We show that a Skoda complex with a general plurisubharmonic weight function is exact if its 'degree' is sufficiently large. This answers a question of Lazarsfeld and implies that not every integrally closed ideal is equal to a multiplier ideal even if we allow general plurisubharmonic weights for the multiplier ideal, extending the result of Lazarsfeld and Lee [LL].

1. Introduction

In complex algebraic geometry, a singular weight function of the form $1/|f|^2 =: e^{-\phi}$, where f is a holomorphic function, plays an important role. It is natural to consider more generally a singular weight $e^{-\phi}$, where ϕ is a plurisubharmonic (psh) function. Given $e^{-\phi}$, there are two fundamental ways to define an ideal sheaf of local holomorphic function germs, say u: collecting those with $|u|^2 e^{-\phi}$ locally bounded above, on the one hand, and collecting those with the local integral $\int_\Omega |u|^2 e^{-\phi}$ finite, on the other hand. The former gives an *integrally closed* ideal and the latter gives a *multiplier* ideal.

A multiplier ideal is always an integrally closed ideal, but the converse had been unknown in a general dimension until [LL] showed the existence of an integrally closed ideal that is not a multiplier ideal of a psh function with analytic singularities (e.g. of the form $\log|f|^2$ for f holomorphic). In this paper, we extend this result to the full generality of multiplier ideals of arbitrary psh functions. The proof of [LL] used the exactness of a *Skoda complex* – a Koszul-type complex of sheaves which involves multiplier ideals in a natural way (see Definition 2.2). For the special case of a psh function with analytic singularities, the exactness is a rather elementary consequence of a local vanishing theorem [L, (9.4.4), (9.6.36)]. However, the general case of the exactness of a Skoda complex cannot be equally shown from the vanishing theorem because of the difficult openness conjecture (3.1) which is not known beyond dimension 2 (see [FJ]). Instead of vanishing, we use the L^2 methods of [Sk72] to a Skoda complex setting and prove the following theorem.

Theorem 1.1. *Let X be a complex manifold, and let L and M be line bundles on X. Let $e^{-\psi}$ be a singular hermitian metric with psh weight for the line bundle M. Let $g_1, \dots, g_p \in H^0(X, L)$. Then there exists an integer $q \geq p$ such that*

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