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On the Representation of Quadratic Forms by Quadratic Forms

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1. Introduction

The study of representing an integral quadratic form by another integral quadratic form has a long history in number theory. In this paper we use matrix notation for quadratic forms, so let $A = (A_{ij})$ and $B = (B_{ij})$ be symmetric positive definite integer matrices of dimensions n and m, respectively. We are interested in finding $n \times m$ integer matrices X such that

$$X^{\mathrm{T}} A X = B, \tag{1}$$

thereby generalizing the classical problem of representing a positive integer as a sum of squares. Although the local-global principle is known to hold for *rational* solutions X of the Diophantine problem (1), existence of solutions over $\mathbb{R}$ (which is here automatic by positive definiteness) and all local rings $\mathbb{Z}_p$ is not enough to ensure the existence of an *integer* solution X. It is therefore natural to look for additional conditions for ensuring that the local-global principle holds also over $\mathbb{Z}$. The usual approach is to fix m, n, and A and then try to represent "large enough" B for dimension m as large as possible in terms of n. In this context, Hsia, Kitaoka, and Kneser [9] have shown the local-global principle to hold whenever $n \geq 2m+3$ and $\min B \geq c_1$ for some constant c_1 depending only on A and n, where (as usual) $\min B$ denotes the first successive minimum of B; that is,

$$\min B = \min_{\mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}} \mathbf{x}^{\mathrm{T}} B \mathbf{x}.$$

Ellenberg and Venkatesh [7] used ergodic theory to show that the condition on n can be greatly improved to $n \geq m + 5$ under the additional assumption that the discriminant of B is square-free. This latter condition has been refined by Schulze-Pillot [15].

The methods just described do not yield any quantitative information about integer solutions to (1). Let $N(A, B)$ denote the number of integer matrices X satisfying (1), and note that this quantity is finite because A is positive definite. Siegel [17] gave an exact formula for a weighted version of $N(A, B)$. Let $\mathfrak{A}$ be a set of representatives of all equivalence classes of forms in the genus of A. For

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