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Veronese Quotient Models of $\overline{\mathrm{M}}_{0,n}$ and Conformal Blocks

Angela Gibney, David Jensen, Han-Bom Moon, & David Swinarski

Introduction

The moduli space of Deligne–Mumford stable n-pointed rational curves, $\overline{\mathrm{M}}_{0,n}$, is a natural compactification of the moduli space of smooth pointed genus 0 curves and has figured prominently in the literature. A central motivating question is to describe other compactifications of $\mathrm{M}_{0,n}$ that receive morphisms from $\overline{\mathrm{M}}_{0,n}$. From the perspective of Mori theory, this is tantamount to describing semi-ample divisors on $\overline{\mathrm{M}}_{0,n}$. This work is concerned with two recent constructions that each yield an abundance of such semi-ample divisors on $\overline{\mathrm{M}}_{0,n}$ and with the relationship between them. The first construction comes from geometric invariant theory (GIT) and the second from conformal field theory.

There are birational models of $\overline{\mathrm{M}}_{0,n}$ obtained via GIT that are moduli spaces of pointed rational normal curves of fixed degree d, where the curves and the marked points are weighted by nonnegative rational numbers $(\gamma, A) = (\gamma, (a_1, \dots, a_n))$ [Gi; GiJM; GiSi]. These *Veronese quotients* $V^d_{\gamma,A}$ are remarkable in that they specialize to nearly every known compactification of $\mathrm{M}_{0,n}$ [GiJM]. There are birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to these GIT quotients, and their natural polarization can be pulled back along this morphism to yield semi-ample divisors $\mathcal{D}_{\gamma,A}$ on $\overline{\mathrm{M}}_{0,n}$.

A second recent development in the birational geometry of $\overline{\mathrm{M}}_{0,n}$ involves divisors that arise from conformal field theory. These divisors are first Chern classes of vector bundles of conformal blocks $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ on the moduli stack $\overline{\mathcal{M}}_{g,n}$. Constructed using the representation theory of affine Lie algebras [Fa; TUY], these vector bundles depend on the choice of a simple Lie algebra $\mathfrak{g}$, a nonnegative integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant integral weights in the Weyl alcove for $\mathfrak{g}$ of level ℓ. For the definition of vector bundles of conformal blocks and related representation-theoretic notations, see Section 4.1. Vector bundles of conformal blocks are globally generated when $g = 0$ [Fa, Lemma 2.5], and their first Chern classes $c_1(\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})) = \mathbb{D}(\mathfrak{g}, \ell, \vec{\lambda})$, the conformal block divisors, are semi-ample.

For $\gamma = 0$, it was shown in [Gi; GiG] that the divisors $\mathcal{D}_{0,A}$ coincide with conformal block divisors for $\mathfrak{sl}_r$ and level 1. Our guiding philosophy is that there is a

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