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Brauer Groups of Singular del Pezzo Surfaces

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1. Introduction

The Brauer group of a variety X, which in this paper we take to mean the cohomology group $\mathrm{Br}\, X = \mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}})$, was extensively studied by Grothendieck [8]. Brauer groups of singular varieties are not particularly well behaved: in particular, the Brauer group of a singular variety need not inject into the Brauer group of its function field. The purely local question, of understanding the Brauer group of the local ring of a singularity, has been well studied: see, for example, [6]. An interesting feature of the results discussed in this paper is that the calculation is a global one and often leads to elements of the Brauer group that are locally trivial in the Zariski topology. One individual example of such an element was given by Ojanguren [13], whose algebra is of order 3 and defined on a singular cubic surface with three A_2 singularities; it will be shown in what follows that this is the only type of singular cubic surface admitting a 3-torsion Brauer element. A more general framework for studying such examples, described by Grothendieck in [8], was developed by De Meyer and Ford [5] to give examples of toric surfaces admitting nontrivial, locally trivial Azumaya algebras.

In this paper we take a slightly different approach that, for varieties with rational singularities, shows how the calculation of the Brauer group can be made very explicit by using the intersection pairing. We then apply this to arguably the simplest interesting class of singular projective surfaces, namely the singular del Pezzo surfaces. These are easy to approach for two reasons: they have rational singularities; and they come with a natural desingularization that is a rational surface. In Proposition 1 we show how to combine the Leray spectral sequence for the desingularization with Lipman's detailed description of the local Picard groups above the singular points [9]. In particular, it follows that the Brauer group may be easily computed using the intersection form on the desingularization. For singular del Pezzo surfaces this is well understood, and in Section 3 we apply Proposition 1 to compute the Brauer groups of all singular del Pezzo surfaces over an algebraically closed field; the Brauer group depends only on the singularity type of the surface. The arguments, and hence the results, are valid in arbitrary characteristic.

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