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On Invariants of Complete Intersections

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Introduction and Background

When R is a local hypersurface (that is, the quotient of a regular local ring by a regular element), Hochster [8, p. 98] defined the invariant θ^R for any pair of finitely generated R-modules M, N such that the localization of N at any prime other than the maximal ideal has finite projective dimension over the corresponding localized ring. This invariant is simply the difference in lengths of two consecutive Tor modules in high enough degree. The vanishing of this invariant is tied to the dimension inequality; to wit, when R is an *admissible* hypersurface and the tensor product of M and N has finite length, $\theta^R(M,N) = 0$ if and only if $\dim M + \dim N \leq \dim R$ [8, Thm. 1.4]. (A ring R is *admissible* if a completion $\hat{R}$ of R at a maximal ideal satisfies $\hat{R} \cong T/(f)$ and if the *dimension inequality, vanishing,* and *positivity* of Serre [14, V.5.1] hold for T; Serre showed that these conditions on T hold when T is a regular local ring containing a field.)

In this paper, we introduce a new invariant, denoted $\boldsymbol{\Theta}_c^R$, in the case that the ring is a standard graded complete intersection with isolated singularity, and we show that if the tensor product of a graded pair M, N has finite length, then $\boldsymbol{\Theta}_c^R(M,N) = 0$ if and only if $\dim M + \dim N \leq \dim R$. Moreover, $\dim M + \dim N \leq \dim R + 1$ regardless of the value of $\boldsymbol{\Theta}_c^R(M,N)$. See Corollary 2.6.

This work continues in the spirit of recent research and takes its motivation from H. Dao, W. F. Moore et al., and Y. Kobayashi. To be specific, Dao [2; 3] provided an in-depth study of θ^R, especially in the case that the ring has an isolated singularity at the maximal ideal, tying the invariant to questions of dimension and rigidity. Along with W. F. Moore, G. Piepmeyer, and M. E. Walker, the author continued this particular study of θ^R under the additional assumption that R is graded and contains a field. In particular, we established the vanishing of θ^R for every pair of finitely generated modules when the dimension of R is even [12, Thm. 3.2], proving, in the graded case, a conjecture posed by Dao [3, Conj. 3.15]. In addition, we showed that θ^R factors through cohomology and gave a formula for the pairing in odd dimension.

Let R be a complete intersection (that is, the quotient of a regular ring Q by a regular sequence of length c). If the $\operatorname{Tor}_j^R(M,N)$ eventually have finite length, then these lengths follow predictable patterns in high degree on even and odd indices.

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