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Quantum Ring of Singularity $X^p + XY^q$

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1. Introduction

Let (X, x) be an isolated complete intersection singularity of dimension $N - 1$. This means that X is isomorphic to the fiber $(f^{-1}(0), 0)$ of an analytic map-germ $f: (\mathbb{C}^{N+k-1}, 0) \to (\mathbb{C}^k, 0)$ and that $x \in X$ is an isolated singular point of X. In particular, if $k = 1$ then (X, x) is called a *hypersurface* singularity. The study of the singularity was initiated by H. Whitney and R. Thom and later developed by V. Arnold, K. Saito, and many other mathematicians from the 1960s and 1980s (see [AGV; He; S; ST]). The classification problem is the central topic in singularity theory. Many geometric and topological invariants were introduced to describe the behavior of the singularity—for example, the Milnor ring, intersection matrix, Gauss–Manin system, and periodic map. Singularity theory has tight connections with many fields of mathematics, including differential equations, function theory, and symplectic geometry.

The papers [FJR1; FJR2; FJR3] construct a quantum theory for a hypersurface singularity given by a nondegenerate quasi-homogeneous polynomial W. The starting point of this research is Witten's work [W] that seeks to generalize the Witten–Kontsevich theorem to the moduli problem of r-spin curves. Unlike in the r-spin case, in which the Witten equation has only trivial solution, in the general W case (e.g., the D_n and E_7 cases) the Witten equation may have nontrivial solutions that cannot be ignored in the construction of the virtual cycle $[\mathcal{W}_{g,k}]^{\text{vir}}$. The Witten equation is defined on an orbifold curve and has the following form:

$$\bar{\partial} u_i + \frac{\overline{\partial W}}{\partial u_i} = 0,$$

where the u_i are sections of appropriate orbifold line bundles.

The Witten equation comes from the study of the Landau–Ginzburg (LG) model in supersymmetric quantum field theory, which can be viewed as a geometrical realization of the $N = 2$ superconformal algebra. The other known model is the nonlinear sigma model corresponding to Gromov–Witten theory. In the LG model, the Lagrangian is totally determined by its superpotential—for instance, a quasi-homogeneous polynomial. There are two possible ways of deriving topological

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