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Topological Symmetry Groups and Mapping Class Groups for Spatial Graphs

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Introduction

By a *graph* we shall mean the underlying space of a finite connected simplicial complex of dimension 1. A *spatial graph* is a graph embedded in a 3-manifold. The theory of spatial graphs is a generalization of classical knot theory. For a spatial graph Γ in S^3, the *mapping class group* $\mathrm{MCG}(S^3, \Gamma)$ (resp., $\mathrm{MCG}_+(S^3, \Gamma)$) is defined as the group of isotopy classes of the self-homeomorphisms (resp., orientation-preserving self-homeomorphisms) of S^3 that preserve Γ setwise. The cardinality of the group describes how many symmetries the spatial graph admits. In [16] it is shown that the group $\mathrm{MCG}(S^3, \Gamma)$ is always finitely presented.

Simon [19] (see also [3; 4] for details) introduced a similar concept, called the *topological symmetry group* of a spatial graph Γ in S^3 and denoted by $\mathrm{TSG}(S^3, \Gamma)$, to describe the symmetries of a spatial graph Γ in S^3. This group is defined as the subgroup of the automorphism group of Γ induced by homeomorphisms of the pair (S^3, Γ). When we allow only orientation-preserving homeomorphisms, we obtain the positive topological symmetry group $\mathrm{TSG}_+(S^3, \Gamma)$.

The aim of this paper is to provide complete answers (Theorems 2.5 and 3.2) to the following question.

Question. When is $\mathrm{TSG}(S^3, \Gamma)$ (resp., $\mathrm{TSG}_+(S^3, \Gamma)$) isomorphic to $\mathrm{MCG}(S^3, \Gamma)$ (resp., $\mathrm{MCG}_+(S^3, \Gamma)$)?

We remark that one of the answers to this question (viz., Theorem 2.5) implies that, if the group $\mathrm{MCG}_+(S^3, \Gamma)$ is finite, then by [3] it is a finite subgroup of $\mathrm{SO}(4)$.

Notation. Let X be a subset of a given polyhedral space Y. Throughout the paper, we denote the interior of X by $\mathrm{Int}\, X$. We will use $N(X; Y)$ to denote a closed regular neighborhood of X in Y. If the ambient space Y is clear from the context, we denote it more briefly by $N(X)$. Let M be a 3-manifold, and let $L \subset M$ be a submanifold with or without boundary. When L is of dimension 1 or 2, we write $E(L) = M \setminus \mathrm{Int}\, N(L)$. When L is of dimension 3, we write $E(L) = M \setminus \mathrm{Int}\, L$.

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