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Characteristic Polynomials, η-Complexes, and Freeness of Tame Arrangements

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1. Introduction

We use notation from Section 2 to state the main result of this paper. Let $\mathcal{A}$ be a central ℓ-arrangement over an arbitrary field $\mathbb{K}$. Fix $H_0 \in \mathcal{A}$, and let $(\mathcal{A}'', m)$ be the Ziegler restriction of $\mathcal{A}$ onto H_0. Let $d\mathcal{A}$ be the deconing of $\mathcal{A}$ with respect to H_0. Let

$$\chi_0(\mathcal{A}, t) = \chi(d\mathcal{A}, t) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} b_{\ell-1-i} t^i$$

be a reduced characteristic polynomial of $\mathcal{A}$, and let

$$\chi(\mathcal{A}'', m, t) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \sigma_{\ell-1-i} t^i$$

be a characteristic polynomial of $(\mathcal{A}'', m)$. Note that $\chi_0(\mathcal{A}, t)$ is defined combinatorially and $\chi(\mathcal{A}'', m)$ algebraically. It is well known that $b_0 = \sigma_0 = 1$ and $b_1 = \sigma_1 = |\mathcal{A}| - 1 = |m|$ (use Theorem 2.3, for example). The inequality $b_2 \geq \sigma_2$ has recently been proved, and the equality $b_2 = \sigma_2$ is closely related to the freeness of $\mathcal{A}$ [2, Thm. 5.1]. This is a generalization of Yoshinaga's freeness criterion for 3-arrangements [13, Thm. 3.2]. Also, it is known that $b_i = \sigma_i$ for $i = 0, 1, \dots, \ell-1$ when $\mathcal{A}$ is a free arrangement (see the proof of Corollary 1.2). Hence it is natural to ask whether $b_i \geq \sigma_i$ holds for $i \geq 3$ and whether or not the equality is related to freeness. In fact, we do not know whether σ_i is nonnegative for $i \geq 3$. In this paper, we assume tameness and give the following answer.

Theorem 1.1. *Let $\mathcal{A}$ be a central ℓ-arrangement. Fix $H_0 \in \mathcal{A}$ and let $(\mathcal{A}'', m)$ be the Ziegler restriction of $\mathcal{A}$ with respect to H_0. If $\mathcal{A}$ and $(\mathcal{A}'', m)$ are both tame, then $b_i \geq \sigma_i \geq 0$ $(i = 0, 1, \dots, \ell-1)$.*

Theorem 1.1 gives a lower bound of $|\chi_0(\mathcal{A}, -1)|$ in terms of $\chi(\mathcal{A}'', m, -1)$; in particular, $|\chi_0(\mathcal{A}, -1)| \geq |\chi(\mathcal{A}'', m, -1)|$. Note that $|\chi_0(\mathcal{A}, -1)|$ is the number of chambers when $\mathbb{K} = \mathbb{R}$. In the category of tame arrangements, then, we say that

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