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Torsion in the Cohomology of Desingularized Fiber Products of Elliptic Surfaces

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0. Introduction

Let W be a smooth projective threefold over an algebraically closed field k. Let l be a prime distinct from $\mathrm{char}(k)$. This paper is concerned with the problem of computing the torsion subgroup $H^3(W, \mathbb{Z}_l)_{\mathrm{tors}}$ of the étale cohomology $H^3(W, \mathbb{Z}_l)$. Before stating the results, we recall five reasons why one is interested in this group.

1. *Relationship to the Brauer group,* $\mathrm{Br}(W)$. The subgroup $H^3(W, \mathbb{Z}_l(1))_{\mathrm{tors}}$ is canonically isomorphic to the l-primary part of the Brauer group modulo its maximal divisible subgroup [Gro, Sec. 8.3].

2. *Birational invariance for smooth projective varieties* [Gro, Thm. 7.4, Thm. 6.1c]. This attribute was used by Artin and Mumford [AMu] to give examples of unirational threefolds that are not rational.

3. *Mirror symmetry.* Take $k = \mathbb{C}$. There is tantalizing empirical evidence (see [BaKr]) that for Calabi–Yau threefolds, $H^3(W(\mathbb{C}), \mathbb{Z})_{\mathrm{tors}}$ should be isomorphic to the first homology of the mirror. No natural isomorphism is currently known.

4. *The integral Tate problem.* Poincaré duality for a smooth projective threefold gives an isomorphism,

$$H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}} \simeq \mathrm{Hom}(H^3(W, \mathbb{Z}_l)_{\mathrm{tors}}, \mathbb{Q}_l/\mathbb{Z}_l(1)).$$

An integral version of the Tate conjecture (or if $k = \mathbb{C}$, of the Hodge conjecture) would imply that $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}}$ is generated by classes of codimension-2 algebraic cycles. The integral Hodge conjecture is known to hold for Fano threefolds [V] (see also [Gra]) and for Calabi–Yau threefolds [V]. It is known to fail for certain threefolds of general type [Ko], although in [Ko] $H^4(W, \mathbb{Z}_l(2))_{\mathrm{tors}} = 0$. Some of the threefolds studied in this paper are birational to Calabi–Yau varieties but most have Kodaira dimension 1, which is unexplored territory.

5. *The Abel–Jacobi map.* The Abel–Jacobi map applies to algebraic cycles that are integrally homologous to zero. A consequence of Theorem 0.3(ii) is that the much-studied complex multiplication cycles have this property (cf. [ST, 3.2]).

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