Michigan Math. J. 62 (2013), 59–79

Quermaßintegrals and Asymptotic Shape of Random Polytopes in an Isotropic Convex Body

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1. Introduction

The aim of this work is to provide new information on the asymptotic shape of the random polytope

$$K_N = \operatorname{conv}\{\pm x_1, \dots, \pm x_N\} \tag{1.1}$$

spanned by N independent random points $x_1, \dots, x_N$ that are uniformly distributed in an isotropic convex body K in $\mathbb{R}^n$. We fix $N > n$ and further exploit the idea of [9] to compare K_N with the L_q-centroid body $Z_q(K)$ of K for $q \simeq \log N$. Recall that the L_q-centroid body $Z_q(K)$ of K has support function

$$h_{Z_q(K)}(x) = \|\langle \cdot, x\rangle\|_q := \left(\int_K |\langle y, x\rangle|^q \, dy\right)^{1/q}; \tag{1.2}$$

background information on isotropic convex bodies and their L_q-centroid bodies is given in Section 2.

This idea has its roots in previous work [11; 19; 22] on the behavior of symmetric random ± 1-polytopes, the absolute convex hulls of random subsets of the discrete cube $D_2^n = \{-1, 1\}^n$. These articles demonstrated that the absolute convex hull $D_N = \operatorname{conv}(\{\pm x_1, \dots, \pm x_N\})$ of N independent random points $x_1, \dots, x_N$ uniformly distributed over D_2^n has extremal behavior—with respect to several geometric parameters—among all ± 1-polytopes with N vertices at every scale of n, where $n < N \leq 2^n$. The main source of this information is the following estimate from [19] (which improves on an analogous result from [11]): for all $N \geq (1 + \delta)n$ (where $\delta > 0$ can be as small as $1/\log n$) and for every $0 < \beta < 1$,

$$D_N \supseteq c\left(\sqrt{\beta \log(N/n)}\, B_2^n \cap Q_n\right) \tag{1.3}$$

with probability greater than $1 - \exp(-c_1 n^\beta N^{1-\beta}) - \exp(-c_2 N)$. Here B_2^n is the Euclidean unit ball and $Q_n = [-1/2, 1/2]^n$ is the unit cube in $\mathbb{R}^n$.

In a sense, the model of D_N corresponds to the study of the geometry of a random polytope spanned by random points that are uniformly distributed in Q_n. Starting from the observation that $Z_q(Q_n) \simeq \sqrt{q}\, B_2^n \cap Q_n$, whence (1.3) can be equivalently written in the form

$$D_N \supseteq c Z_{\beta \log(N/n)}(Q_n), \tag{1.4}$$

Received September 27, 2011. Revision received October 7, 2012.