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Determinantal Facet Ideals

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Introduction

Let K be a field, $X = (x_{ij})$ an $m \times n$ matrix of indeterminates, and $S = K[X]$ the polynomial ring over K in the indeterminates x_{ij}. We assume that $m \leq n$. Classically the ideals $I_t(X)$ generated by all t-minors of X have been considered. Hochster and Eagon [15] proved that the rings $S/I_t(X)$ are normal Cohen–Macaulay domains. A standard reference on the classical theory of determinantal ideals, including the study of the powers of $I_t(X)$, is the book by Bruns and Vetter [4]. In addition, the study of a more general class of ladder determinantal ideals has been motivated by geometrical considerations [6]. A new aspect to the theory of determinantal ideals was introduced by Sturmfels [17] and Caniglia et al. [5], who showed that the t-minors of X form a Gröbner basis of $I_t(X)$ with respect to any monomial order that selects the diagonals of the minors as leading terms. This technique yields a new proof of the Cohen–Macaulayness of the determinantal rings $S/I_t(X)$ and was subsequently used also to compute important numerical invariants of these rings—including the a-invariant, the multiplicity, and the Hilbert function (see [2; 7; 13]). Bruns and Conca [1] have written an excellent survey on the theory of determinantal ideals with regard to the Gröbner basis aspect that includes many references to more recent work.

Applications in algebraic statistics prompted the study of determinantal ideals generated by quite general classes of minors, including ideals generated by adjacent 2-minors [11; 16] or ideals generated by an arbitrary set of 2-minors in a $2 \times n$ matrix [12]. Thus one may raise the following questions. Given an arbitrary set of minors of X, what can be said about the ideal they generate? When is such an ideal a radical ideal, and when is it a prime ideal? What is its primary decomposition, when is it Cohen–Macaulay, and what is its Gröbner basis? Apart from the classical cases mentioned before, satisfactory answers to some of these questions are known for ideals generated by arbitrary sets of 2-minors of a $2 \times n$

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