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Bulk Universality Holds Pointwise in the Mean for Compactly Supported Measures

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1. Introduction

Let μ be a finite positive Borel measure with compact support and infinitely many points in the support. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \dots$, satisfying the orthonormality conditions

$$\int p_j p_k \, d\mu = \delta_{jk}.$$

Throughout we use μ' to denote the Radon–Nikodym derivative of μ. The nth reproducing kernel for μ is

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y), \tag{1.1}$$

and the normalized kernel is

$$\tilde{K}_n(x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y). \tag{1.2}$$

In the theory of n-by-n random Hermitian matrices (the so-called unitary case), there arise probability distributions on the eigenvalues that are expressible as determinants of reproducing kernels [5, p. 112]:

$$P^{(n)}(x_1, x_2, \dots, x_n) = \frac{1}{n!} \det(\tilde{K}_n(x_i, x_j))_{1 \le i, j \le n}.$$

One may use this to compute a host of statistical quantities—for example, the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One important quantity is the m-point correlation function for $\mathcal{M}(n)$ [5, p. 112]:

$$\begin{aligned} &R_m(x_1, x_2, \dots, x_m) \\ &\quad = \frac{n!}{(n-m)!} \int \cdots \int P^{(n)}(x_1, x_2, \dots, x_n) \, dx_{m+1} \, dx_{m+2} \dots dx_n \\ &\quad = \det(\tilde{K}_n(x_i, x_j))_{1 \le i, j \le m}. \end{aligned}$$

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