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Moduli Spaces of Curves and Cox Rings

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1. Introduction

Let k be a field, X a smooth projective geometrically irreducible k-variety, and $\mathscr{C}$ a smooth projective geometrically irreducible k-curve of genus $g_{\mathscr{C}}$. Let $\mathscr{K}_X$ be the canonical class of X. For every element y of the dual $\mathrm{NS}(X)^\vee$ of the Néron–Severi group of X, let $\mathrm{Mor}(\mathscr{C}, X, y)$ denote the quasi-projective k-variety parameterizing the morphisms $f : \mathscr{C} \to X$ such that $[f_*\mathscr{C}] = y$. By [De, Sec. 2.11], every irreducible component of $\mathrm{Mor}(\mathscr{C}, X, y)$ has dimension at least $(1 - g_{\mathscr{C}}) \dim(X) + \langle y, -\mathscr{K}_X \rangle$. The latter quantity will be referred to as the *expected dimension* of $\mathrm{Mor}(\mathscr{C}, X, y)$. It is a natural though difficult question to ask for the number and dimension of the irreducible components of $\mathrm{Mor}(\mathscr{C}, X, y)$. Works addressing this question for specific families of varieties include [Ca; CS; dJS; HRS1; HRS2; KLO; KP; Pe; T].

In this paper we study the question using the so-called Cox ring of X, restricting ourselves to a class of varieties whose Cox ring has an especially simple presentation. It is known, at least when $\mathscr{C}$ is rational, that the Cox ring of a toric variety X provides a useful description of the moduli spaces $\mathrm{Mor}(\mathscr{C}, X, y)$ [Ba; Bo3; G]. Toric varieties may be characterized by the fact that their Cox ring is a polynomial ring; hence they are the simplest varieties from the viewpoint of the description of the Cox ring.

Here we consider varieties whose Cox ring may be presented by only one equation, which has moreover a kind of linearity property with respect to a certain subset of variables (see Definitions and Notation 2.1 for more precision). We will call such varieties *linear intrinsic hypersurfaces* (the terminology *intrinsic hypersurface* is borrowed from [BHau]). Let $\mathrm{Mor}(\mathscr{C}, X, y)^\circ$ denote the open set of $\mathrm{Mor}(\mathscr{C}, X, y)$ consisting of those morphisms that do not factor through the boundary—in other words, the union of the divisors of the sections used to present the Cox ring.

Our main result reads as follows (see Theorem 2.4 for a more precise statement).

Theorem 1.1. *Let X be smooth projective $\mathbf{Q}$-variety that is a linear intrinsic hypersurface. Assume that certain rational combinatoric series derived from the equation of the Cox ring fulfill some explicit analytic properties. Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve. For every $y \in \mathrm{NS}(X)^\vee$ lying*

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