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# Continuous Closure of Sheaves

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Definition 1. Let $I = (f_1, \dots, f_r) \subset \mathbb{C}[z_1, \dots, z_n]$ be an ideal. Following [Br], a polynomial $g(z_1, \dots, z_n)$ is in the *continuous closure* of $I$ if and only if there are continuous functions $\phi_i$ such that $g = \phi_1 f_1 + \cdots + \phi_r f_r$. These polynomials form an ideal $I^C \supset I$. For example,

$$z_1^2 z_2^2 = \frac{\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2} z_1^3 + \frac{\bar{z}_2 z_1^2}{|z_1|^2 + |z_2|^2} z_2^3$$

shows that $z_1^2 z_2^2 \in (z_1^3, z_2^3)^C \setminus (z_1^3, z_2^3)$.

Definition 1 is very natural, but it is not clear that it gives an algebraic notion (since $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ does not map continuous functions to continuous functions) or that it defines a sheaf in the Zariski topology (since a continuous function may grow faster than any polynomial).

This paper has three aims.

- We give a purely algebraic construction of the continuous closure of any torsion-free coherent sheaf (Definition 6). Although the construction makes sense for any reduced scheme, even in positive and mixed characteristic, it is not clear that it corresponds to a more intuitive version in general.
- In characteristic 0 we prove that one gets the same definition of $I^C$ using various subclasses of continuous functions (Corollary 19).
- We show that taking continuous closure commutes with flat morphisms whose fibers are seminormal (Corollary 21), at least in characteristic 0. In particular, the continuous closure of a coherent ideal sheaf is again a coherent ideal sheaf (both in the Zariski and in the étale topologies) and it commutes with field extensions.

It should be noted that, although our definition of the continuous closure is purely algebraic and without any reference to continuity, the proof of these base change properties uses continuous functions in an essential way.

Instead of working with $\mathbb{C}$ or other algebraically closed fields, one can also define the continuous closure over any topological field. The most interesting is the real case, considered in [FK]. The answer turns out to be quite different; for instance, over $\mathbb{C}$ the continuous closure of $(x^2 + y^2)$ is itself but over $\mathbb{R}$ it is the

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