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# Inversion Invariant Bilipschitz Homogeneity

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## 1. Introduction

This paper examines metric spaces that are bilipschitz homogeneous and remain so after they are inverted (see Section 2 for definitions). The general idea is that, in such spaces, the metric doubling property can be improved to Ahlfors $Q$-regularity and local connectedness can be improved to linear local connectedness.

Bilipschitz homogeneous Jordan curves have been well studied (see e.g. [Bi; GH2; HM; M1; R]). Progress has also been made in the study of (locally) bilipschitz homogeneous geodesic surfaces (see [L]). This paper focuses on the stronger assumption of inversion invariant bilipschitz homogeneity in the context of more general doubling metric spaces. Our main results are as follows.

Theorem 1.1. *Let $L, D \geq 1$. Suppose $X$ is a proper, connected, and $D$-doubling metric space. If there exists a $p \in X$ such that both $X$ and the inversion of $X$ at $p$ are $L$-bilipschitz homogeneous then $X$ is $Q$-regular, with regularity constant depending only on $D$ and $L$.*

Theorem 1.2. *Suppose $X$ is a proper, connected, and locally connected doubling metric space. If there exists a $p \in X$ such that both $X$ and the inversion of $X$ at $p$ are uniformly bilipschitz homogeneous, then $X$ is $LLC_1$. If, in addition, we assume that $X$ has no cut points, then $X$ is also $LLC_2$.*

We remark that Theorem 1.2 is qualitative, not quantitative, in nature. It would be interesting to know if a quantitative result is possible.

Before proceeding into the body of the paper, we discuss a few immediate consequences of these two theorems. For one, these results allow us to recover a stronger version of [F1, Thm. 1.2] in which the $LLC_1$ condition (i.e., bounded turning) need not be assumed (see also [F1, Thm. 1.1]).

Corollary 1.3. *Let $\Gamma$ denote a Jordan curve in $\mathbb{R}^n$. The curve $\Gamma$ is an Ahlfors $Q$-regular quasicircle if and only if there exists a point $p \in \Gamma$ such that both $\Gamma$ and the Euclidean inversion of $\Gamma$ at $p$ are uniformly bilipschitz homogeneous.*

The sufficiency follows from Theorem 1.1 and Theorem 1.2. The necessity follows from the fact that an $LLC_1$ and Alhfors $Q$-regular Jordan curve in $\mathbb{R}^n$ is bilipschitz

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